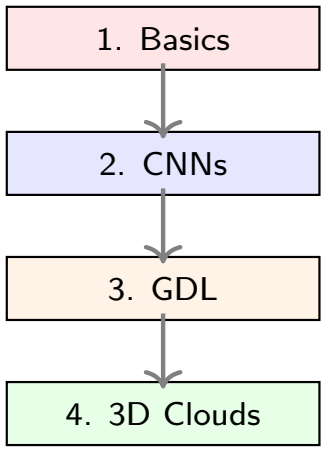
Geometric Deep Learning

Sotirios Nikoloutsopoulos

Presentation Outline

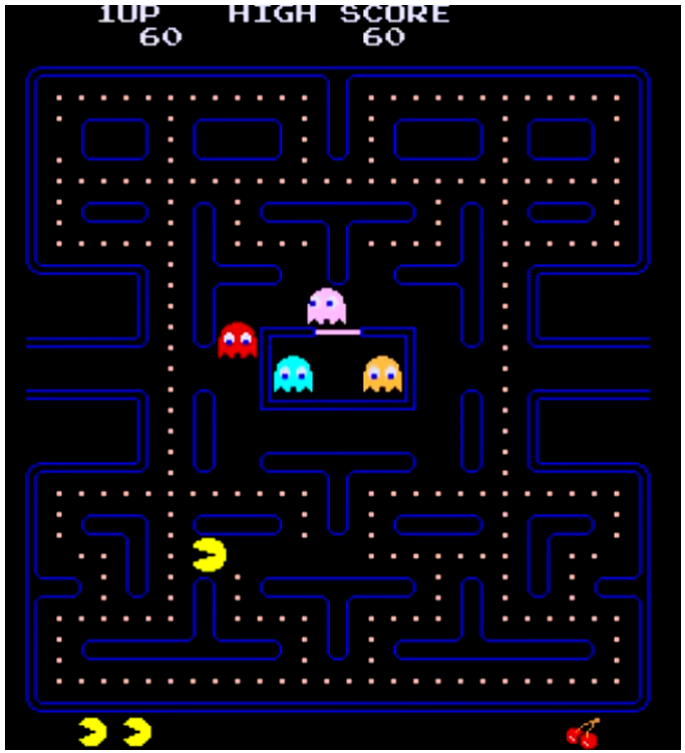
1. **The Big Picture**
   - What is a Neural Network?
   - Memorization: Why raw Pixels are not enough
2. **Classic Vision (CNNs)**
   - How Convolutions work
   - Rotation & Scaling challenges
3. **Geometric Deep Learning (GDL)**
   - Symmetry & Equivariance
   - Steerable Filters
4. **3D Point Clouds**
   - Relational Reasoning (Attention)
   - The Encoder-Decoder Logic



The Big Picture: What is a Neural Network?

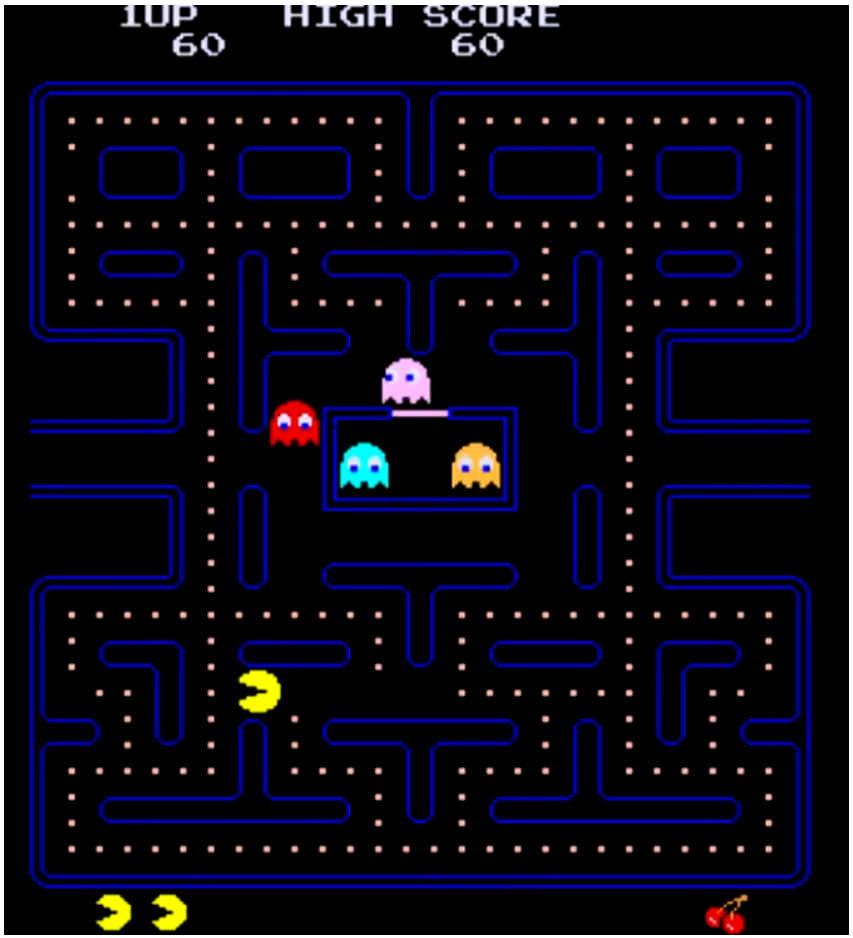
Think of a **Neural Network** as a **function** that takes an **input** and makes a **prediction**.

- **Input:**
  An image (like Pac-Man).
- **The Task:**
  The function should tell us what to do next.



What does the Function learn?

To make a prediction, the function must **see** and **understand** the components of the game.



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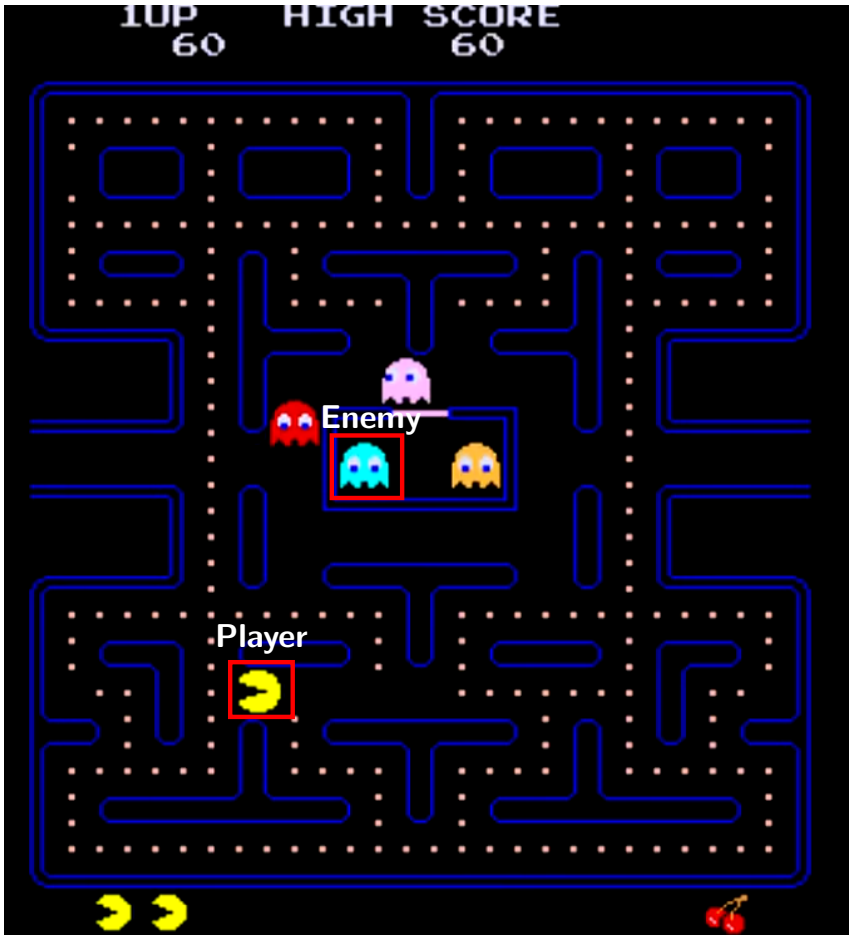
  Identifying objects. *"Is this a Ghost or a Wall?"*



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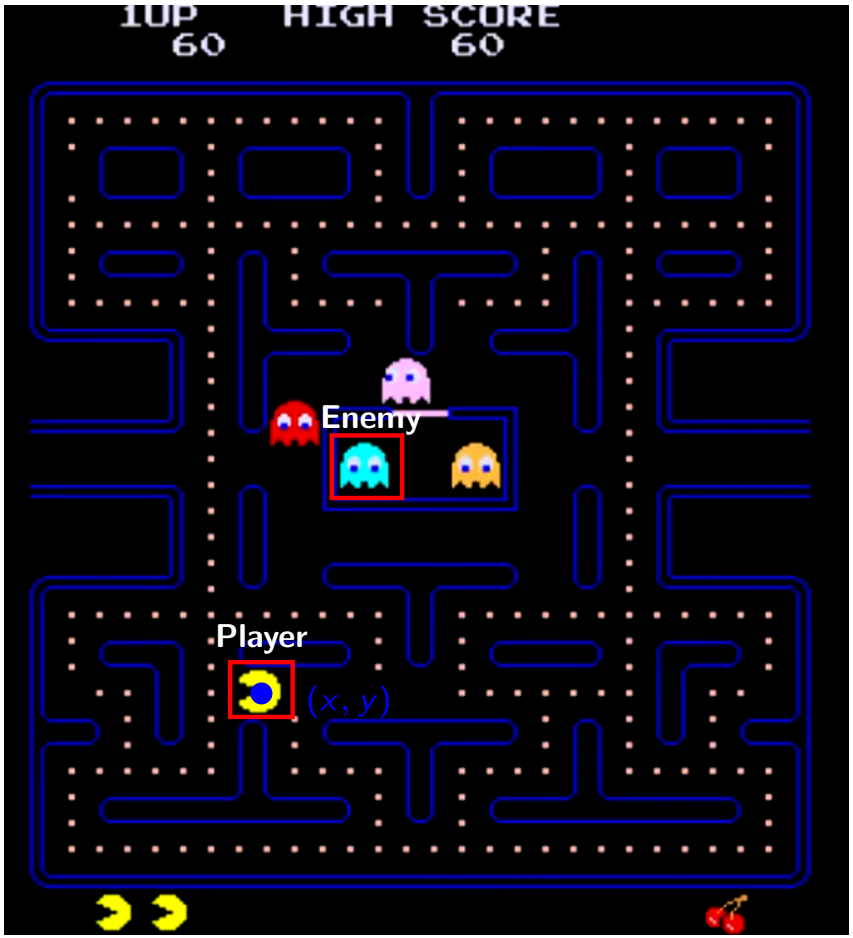
- **Regression:**
  Finding positions. *"Where exactly is Pac-Man?*
  $(x, y) = (12, 45)$



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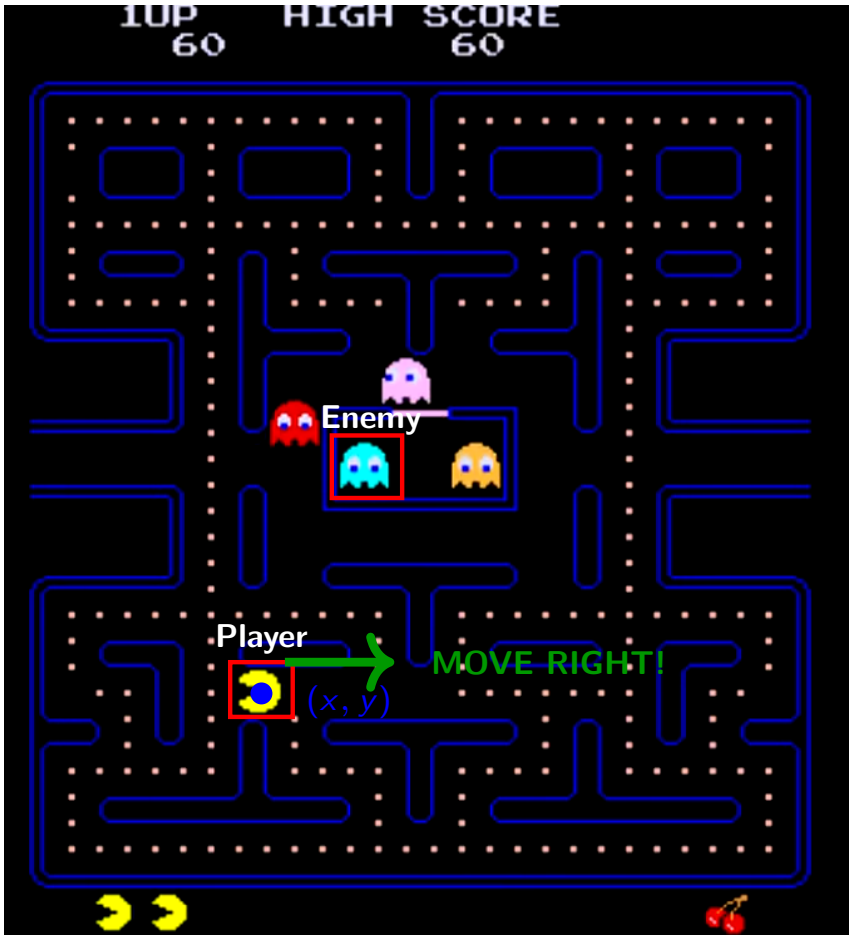
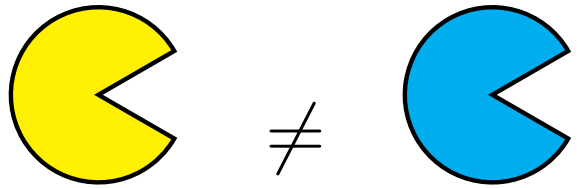
- **Control / Policy:**
  Making a decision. *"Ghost is near → Move Up!"*



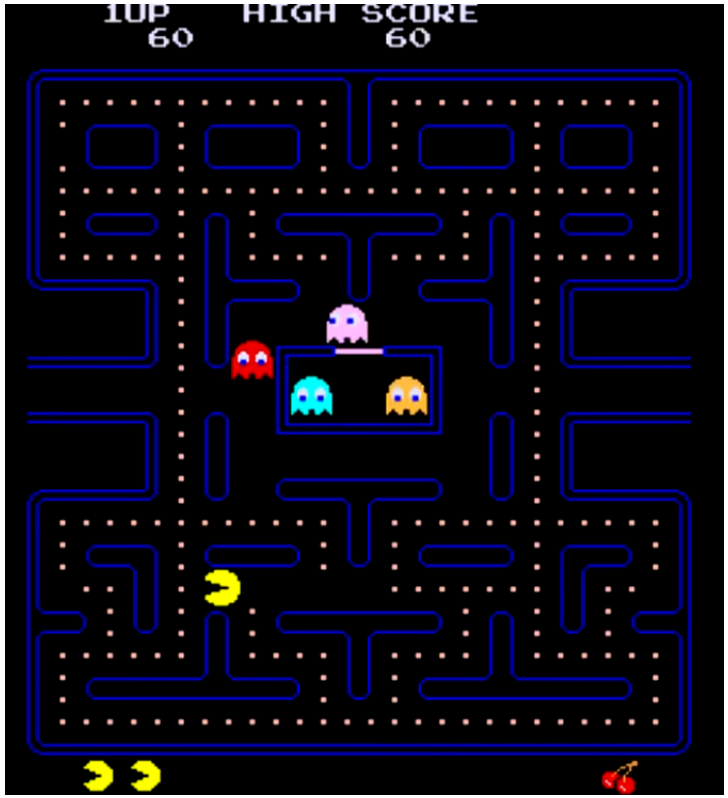
Memorization: Why raw Pixels are not enough

The Challenge:
Change the player's color from **Yellow** to **Blue**.



Training Sample
RGB: (255, 255, 0)

Testing Sample
RGB: (0, 255, 255)



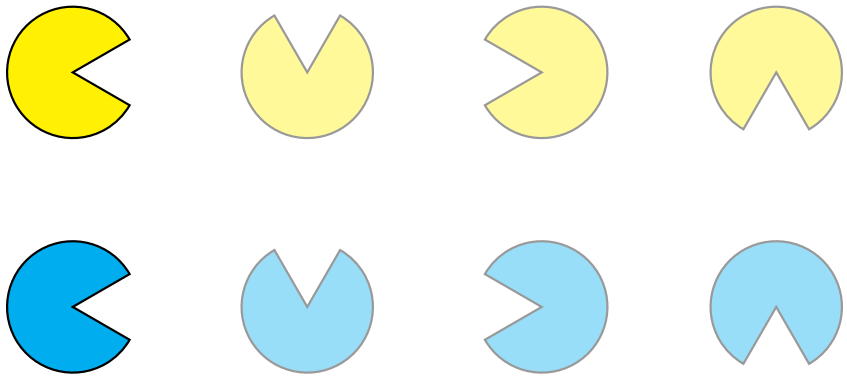
- **Human:** It's just Pac-Man in blue!
- **Model:** `Wait 🧐 ... blue pixels? I know this!` `Prediction: GHOST (99% Confidence)`

The Logic Gap

The model hasn't learned the **geometry** (shape); it has only memorized a specific **color pattern**.

The Cost of Orientation

Normally, we must learn every direction for every color individually.



But do we really need to learn every possible angle from scratch?

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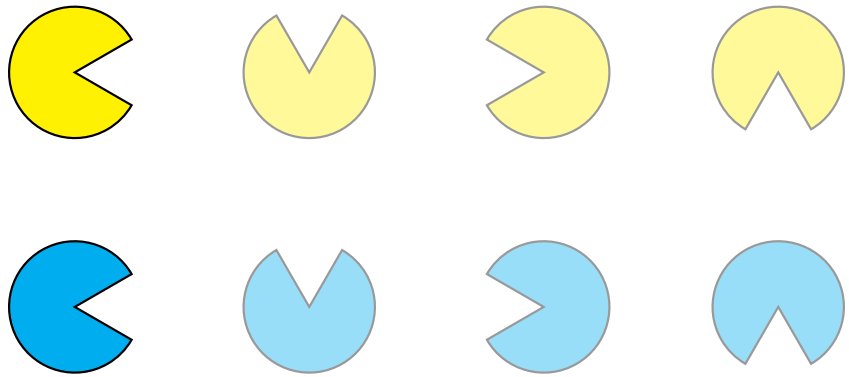


But do we really need to learn every possible angle from scratch?

Option A: Augmentation

- Training with multiples of 90° rotations to **augment** the dataset.

Option B: Geometric Prior

- **Hard-code rotation** laws directly into the **model**.
  $\Rightarrow$ **Data efficiency!**.

Model Architecture

In the next slides, we will analyze the visual component of the model.

Convolutional Neural Networks (CNNs)

From pixels to prediction:

$\Longrightarrow$

Input Image (I)

→ **Conv Layer** → Pattern Detection (output pixels)

→ **Pooling Layer** → Aggregation op (Max, Avg, Min)

→ **Flatten** → Vectorization 2D to 1D

→ **Prediction** → Final Decision

- **Convolution:** **Detect patterns** (edges, shapes)
  - Scan small local pixel **neighborhoods**.
- **Pooling:** **Reduce dimensions**
  - Keep only the most **important information**.
- **Flatten:** **Vectorize features**
  - Transform the 2D grid into a **1D vector**.

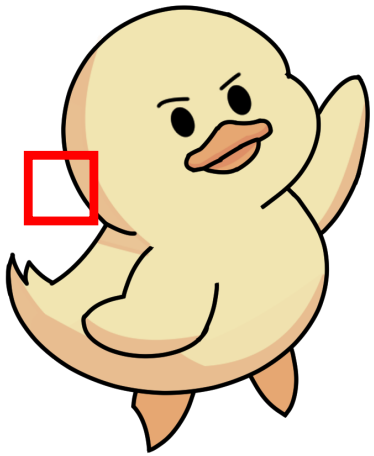
Convolution: The Visual Intuition

The Process: A **Kernel** (Filter) scans the image to find features.

Feature Map (M)

Score		
		...
	...	

Input Image (I)



Scan →

Kernel (K)

3×3		

What's happening?

For every position, we calculate a **Similarity Score**:

1. Overlay the Kernel on a part of the Image.
2. Multiply matching pixels.
3. Sum them up → Put number in M.

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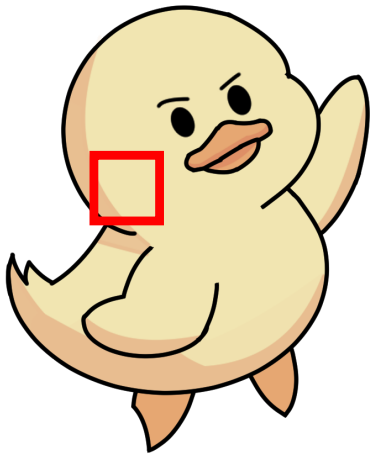

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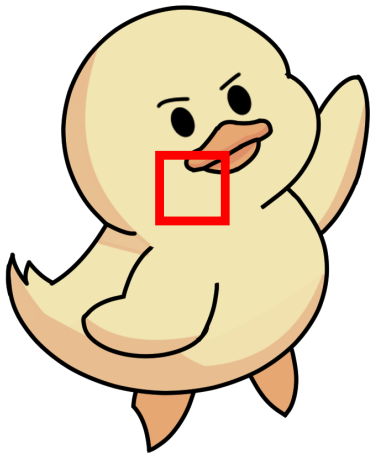

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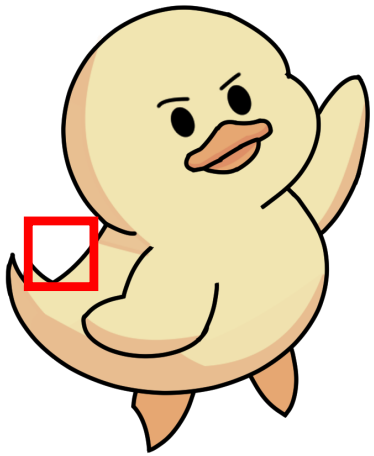

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Score	Score	$\cdots$
	$\cdots$	

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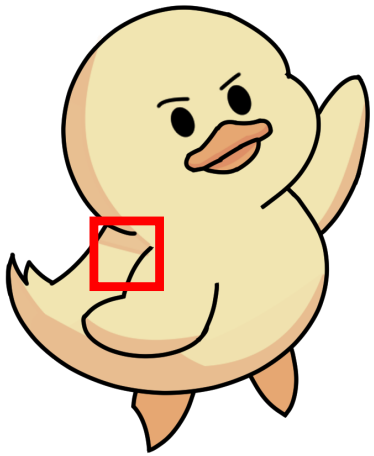

Scan $\longrightarrow$

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1. Overlay the Kernel on a part of the Image.
2. Multiply matching pixels.
3. Sum them up $\rightarrow$ Put number in M.

The Architecture Solution: Convolution

The Process: The Kernel K slides over Image I to compute the Feature Map M.

Input I (with 3×3 sliding window)

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Window

$*$

Kernel K (3×3)

1	2	3
4	5	6
7	8	9

Full Feature Map M

45	...	...
	...	

The Calculation

Dot Product (Element-wise)

$$M_{1,1} = \sum (I_{window} \odot K)$$

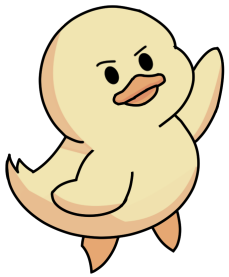
$$(1 \cdot 1) + (1 \cdot 2) + \cdots + (1 \cdot 9) = 45$$

A few kernel examples

1. Blurred (Smoothing)

$$K_{blur} = \frac{1}{15^2} \underbrace{\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}}_{15 \times 15}$$

2. Original Data

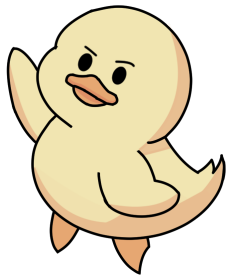


$$\text{Identity } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. Rotated 90°

$$M_{rot} = \begin{pmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{pmatrix}$$

4. Flipped (Reflection)



$$M_{flip} = \begin{pmatrix} -1 & 0 & \text{cols} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x + \text{cols} \\ y \end{pmatrix}$$

The Bridge: From CNNs to Geometry

CNNs learn filters, but are only **translation invariant**.

The Challenge:

- **Rotation (θ):** An object can be at any angle.
- **Scale (s):** In cameras, objects change size based on distance.
- *So, how do we actually design filters to handle every possible size and angle?*

The solution is **Geometric Deep Learning (GDL)**:

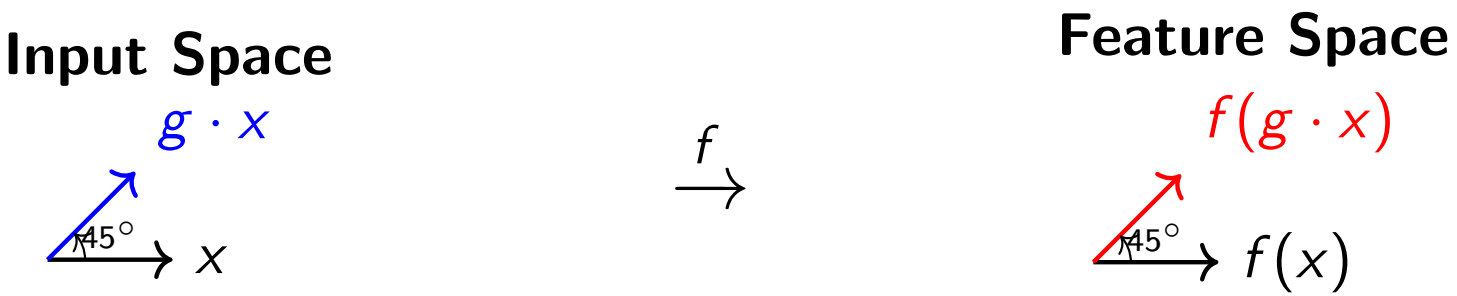
- It's our **Rulebook**.
- It allows the model to use **actions** of geometry
  - Rotation, Scale, Symmetry.

The Formal Language: Equivariance

In **Geometric Deep Learning**, the actions $\{g\}$ are **Equivariant**.

What does $f(g \cdot x) = g \cdot f(x)$ mean?

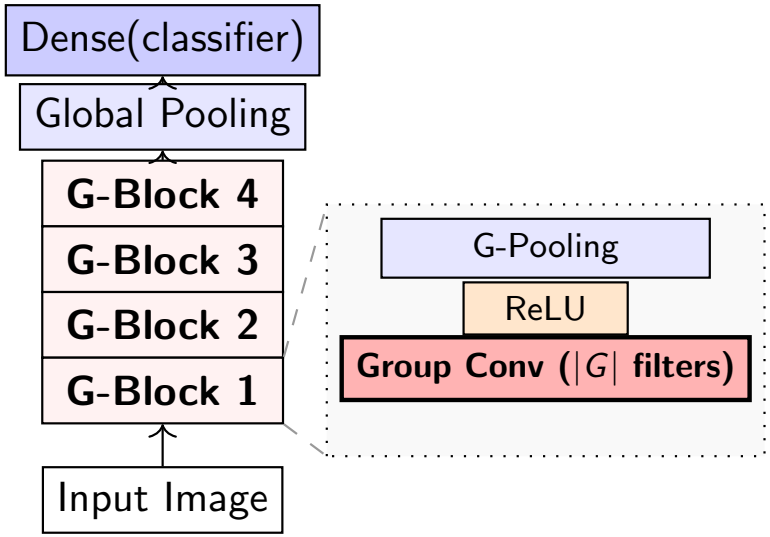
If we **rotate** the input (x) by an angle (g), the output features should **rotate** by the same angle.



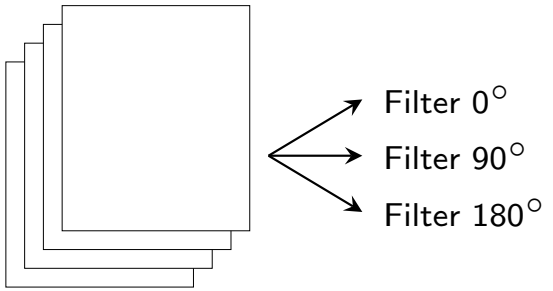
Group Equivariant CNNs (G-CNNs): Stacked Filters

CNN Architecture

How do we build an equivariant layer? $f(g \cdot x) = g \cdot f(x)$

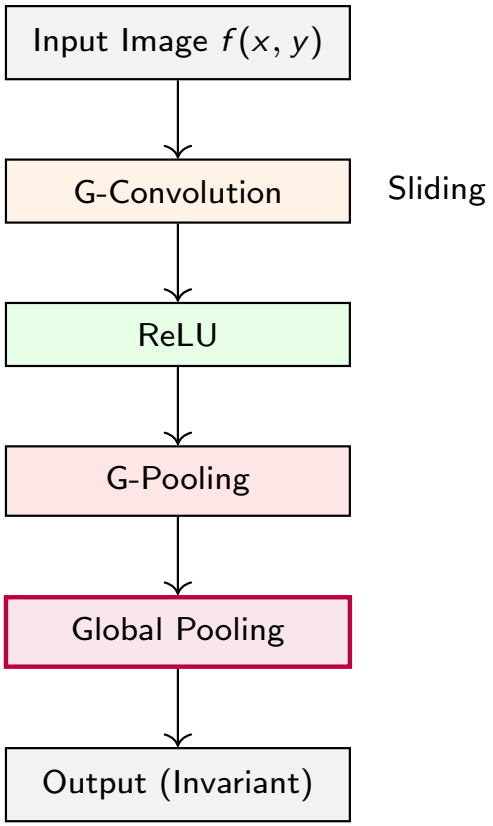


Group of Filters



$$\mathcal{F}_{\mathrm{G}} = \{\mathrm{Filter}_\theta \mid \theta \in \{0^\circ, 90^\circ, \dots\}\}$$

G-CNNs: Pixel-to-Pixel Logic



G-Convolution (Sliding + Rotation)

- Standard **sliding window** logic.
- Computes features for all rotations $s \in G$.

Maps $f(x, y)$ to a 3D-map $f(x, y, s)$.

G-Pooling (Pose Invariance)

- At each pixel (x, y), picks the best rotation s:

$$\text{Output}(x, y) = \max_{s \in G} f(x, y, s).$$

- It **forgets** the **rotation** but keeps the feature.

Global Pooling (Final Decision)

- Aggregates **all** positions (x, y) into **a single scalar value**.
- **Classification Tip:** In the final layer omit G-pooling. Use N filters and perform Global Average to get one number per **class**.

Steerable filters

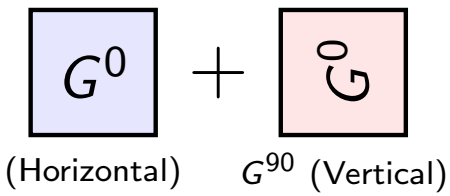
We learn a set of basis:

- E.g. **two basis kernels**: G^0 (horizontal) and G^{90} (vertical).

The Steerable Equation

Any orientation θ can be synthesized as a linear combination:

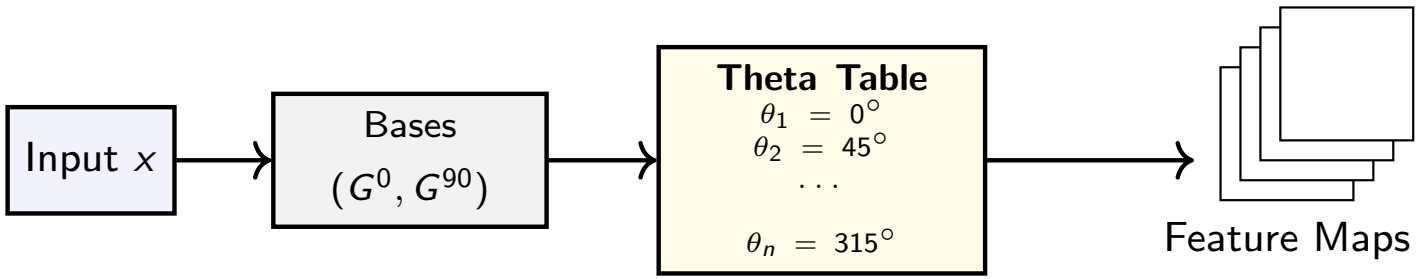
$$G^{\theta} = \cos(\theta) \cdot G^0 + \sin(\theta) \cdot G^{90}$$



- **Parameters:** We only store the weights for G^0 and G^{90}.
- **Inductive Bias:** The model is **forced** to follow the laws of rotation.

The Orientation Table: Learning Angles

Instead of a fixed grid, we maintain a **Table of Orientations** $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ as trainable parameters.



Gradient-Based Optimization

Since the synthesis $G^\theta = \cos(\theta)G^0 + \sin(\theta)G^{90}$ is **differentiable**, we can update the entire table via gradient-based optimizers (e.g. compute derivatives):

$$\theta_{new} \leftarrow \theta - \eta \frac{\partial \mathcal{L}}{\partial \theta} \quad \Longrightarrow \quad \text{Optimizes the } \textbf{Looking Angle}$$

How to Scale: The Inverse Mapping

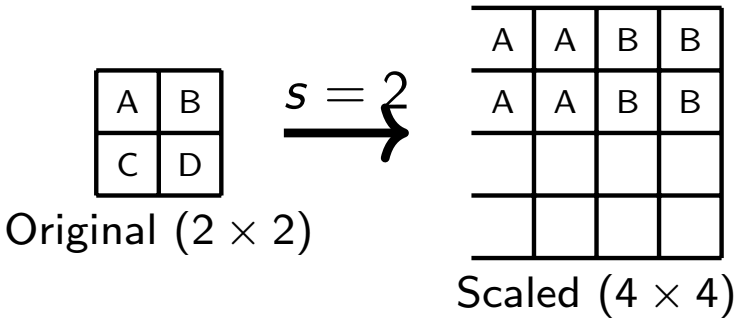
To scale a filter by a factor s, we calculate the new value for every pixel (x, y) using the **Inverse Map**:

The Rule

$$G_{new}(x, y) = G_{old}\left(\frac{x}{s}, \frac{y}{s}\right)$$

Step-by-Step Example ($s = 2$):

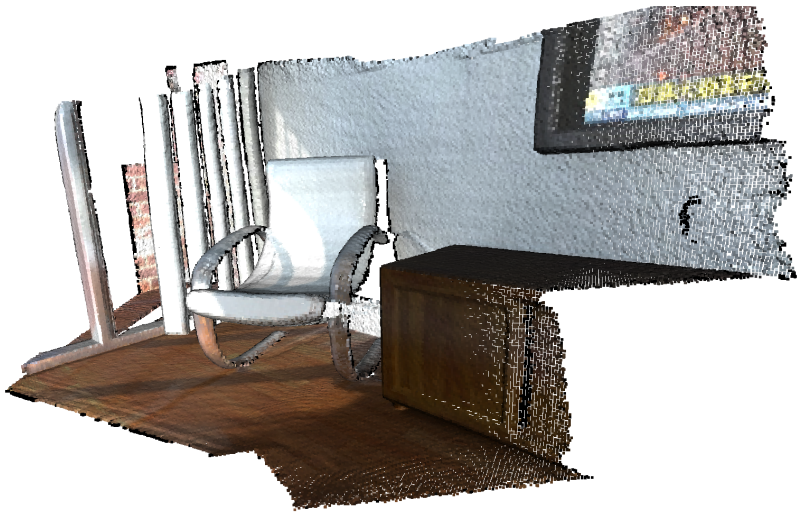
1. We want to fill the pixel at $(x', y') = (2, 2)$ in the **large** filter.
2. We look at the **original** filter at position $(\frac{2}{2}, \frac{2}{2}) = (1, 1)$.
3. We copy that value to the new position.



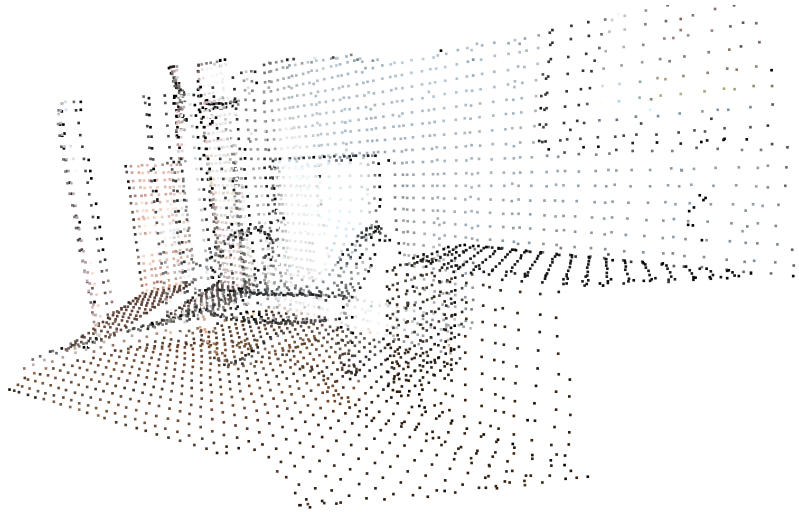
Original (2×2) Scaled (4×4)

Practical Use: 3D Point Clouds

A 3D Point Cloud is just a collection of coordinates (x, y, z) from a LiDAR or 3D scanner.



Point Cloud with RGB

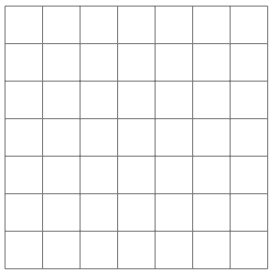


Geometry only

- **Unordered Set:** Unlike pixels, there is no 1st, 2nd, . . . , point.
- **Permutation Invariance:** The model must produce the same result regardless of point ordering.
- **Continuous:** Points can be anywhere, not just on a grid.

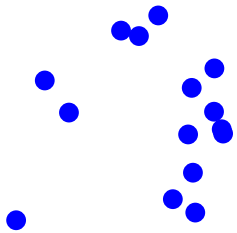
Why CNNs Fail on 3D Point Clouds

- **CNNs** need a **Grid** (like pixels in a 2D image).
- **Point Clouds** have no grid, no rows, and no columns.



CNN: Fixed Grid

$\neq$



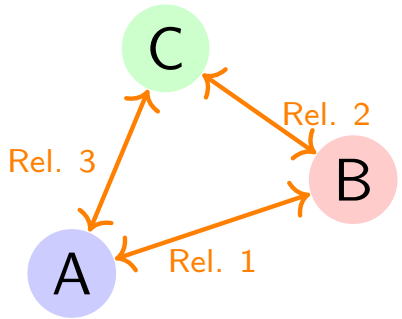
Point Cloud: Floating Points

Solution: Instead of a **sliding window**, we use **Attention**. Points relate to one another directly to identify geometric structures.

Attention: A Tool for Learning Relations

In our 3D world, we use **attention** to achieve **Relational Reasoning**.

- **Standard AI:** Treats points as **isolated coordinates**.
- **Relational AI (attention):** Analyzes the **interactions** between points.

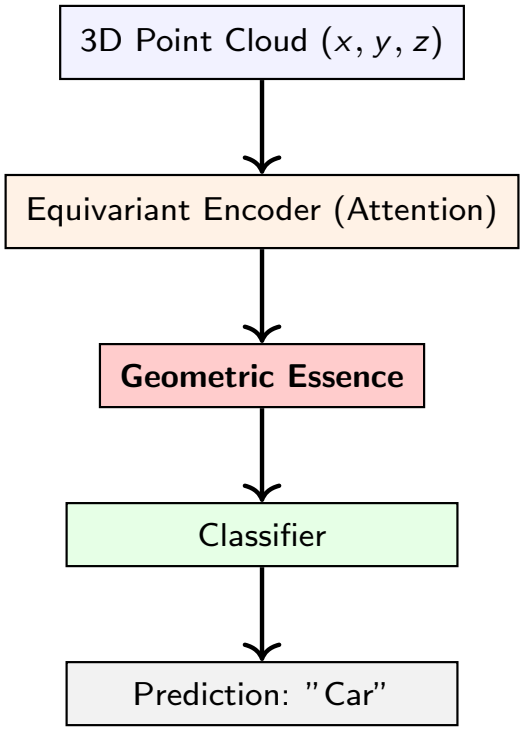


What is a Relation?

- **Distance** between points.
- **Relative Angles**.

Key Idea: Instead of learning (x, y, z), the model learns the **distances** between points. Since distances don't change when things rotate, the model is **invariant**!

Putting it all together: Encoder-Decoder



How it works:

1. **Encoder:** Uses Attention to analyze through points and find relations.
2. **Bottleneck:** Only the **Geometric Essence** (the shape) survives.
3. **Invariance:** Since the model focuses on **distances** (which never change) rather than **coordinates** (which change during rotation), the final prediction remains the same!

Summary Takeaways

- **Geometry is a Prior:** By encoding symmetry, we give the network common sense about the physical world.
- **Data Efficiency:** Mathematical constraints reduce the need for massive datasets.
- **Universal Framework:** GDL provides a unified language for CNNs, Graphs, and 3D data.

The End

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Useful links

- Group Equivariant Convolutional Networks
- Learning Steerable Filters for Rotation Equivariant CNNs
- Group Invariant Global Pooling