Cuntz's Rose

A First Glimpse at Graph Algebras

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Definition

A Hilbert space $\mathcal{H}$ is an inner product space, complete with the norm arising from the inner product.

- Our spaces will be vector spaces over the complex numbers $\mathbb{C}$.
- Our inner products will be conjugate linear in the first variable.

We recall

Definition

An inner product on a complex vector space X is a positive hermitian sesquilinear form $\sigma : X \times X \to \mathbb{C}$.

In our case, $\sigma(\lambda x_1 + x_2, y) = \overline{\lambda}\sigma(x_1, y) + \sigma(x_2, y)$.

In a Hilbert space $\mathcal{H}$ we define linear operators $T : \mathcal{H} \to \mathcal{H}$ satisfying,

$$T(x + \lambda y) = T(x) + \lambda T(y)$$

and we call then bounded when,

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} \ : \ x \in \mathcal{H},\ x \neq 0\right\} < +\infty$$

As it turns out, this is equivalent to the operator being (Lipschitz) continuous. What's more, $\|\cdot\|$ as defined, is a norm on the set of bounded operators.

We denote $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on a Hilbert space $\mathcal{H}$ as presented.

Definition

We call an operator $T \in \mathcal{B}(\mathcal{H})$ an isometry when for all $x \in \mathcal{H}$.

$$\|Tx\| = \|x\|$$

Then, $\|T\| = 1$ for any isometry T. The operator $Tx = x$, denoted by I, is the simplest case.

An important theorem in the theory of Hilbert spaces is the following,

Theorem (Riesz-Fréchet Representation Theorem)

For every linear functional $\pi \in \mathcal{H}^*$, there exists a unique vector $y \in \mathcal{H}$ such that $\pi(x) = \langle y, x\rangle$ for all $x \in \mathcal{H}$. What's more, $\|\pi\| = \|y\|$.

With the help of the Riesz-Fréchet representation theorem, we can prove that for each $T \in \mathcal{B}(\mathcal{H})$ there exists a unique map $T^* \in \mathcal{B}(\mathcal{H})$ such that,

$$\langle x, Ty \rangle = \langle T^* x, y \rangle$$

for all $x, y \in \mathcal{H}$. Then, T^* is called the adjoint of T.

For two $T,\ S \in \mathcal{B}(\mathcal{H})$ and for $\lambda \in \mathbb{C}$,

- $T^{**} = T$
- $(TS)^* = S^* T^*$
- $(\lambda T + S)^* = \overline{\lambda} T^* + S^*$
- $\|T^*\| = \|T\|$
- $\|T^* T\| = \|T\|^2$

With that in mind and with the help of the polarization identity,

$$\langle x, Ty\rangle = \frac{1}{4}\sum_{k=0}^{3}(-i)^k\langle x + i^k y, T(x + i^k y)\rangle$$

for $T \in \mathcal{B}(\mathcal{H})$, we can prove the following,

Proposition

An operator $T \in \mathcal{B}(\mathcal{H})$ is an isometry if and only if $T^*T = I$.

At this point we note that $\mathcal{B}(\mathcal{H})$ is a complete space and that the operation defined by $^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called an involution. This is a specific example of an involution, as we will shortly note.

We need two kinds of operators to be ready for graph algebras,

Definition

Let $T \in \mathcal{B}(\mathcal{H})$. We say that T is a partial isometry if T is an isometry on $\ker T^{\perp}$.

Definition

Let $T \in \mathcal{B}(\mathcal{H})$. We say that T is a projection onto the closed subspace $T(\mathcal{H})$ if $T^* = T^2 = T$.

To be more specific, in the theory of Hilbert spaces, we may decompose $\mathcal{H}$ into a direct sum of closed subspaces $\mathcal{M}_1, \ldots, \mathcal{M}_n$. Then, each element $x \in \mathcal{H}$ can be decomposed as,

$$x = x_1 \oplus \cdots \oplus x_n$$

The projections onto $\mathcal{M}_j$ are defined via $P_j x = x_j$.

We build this, definition upon definition, and compare with $\mathcal{B}(\mathcal{H})$.

Definition

An algebra A is a vector space equipped with a bilinear map $\cdot : A \times A \to A$ such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. We usually omit the $\cdot$ and write ab for $a \cdot b$.

Observe that composition in $\mathcal{B}(\mathcal{H})$ is such an operation.

Definition

A norm $\|\cdot\|$ on A is called submultiplicative, if $\|ab\| \leq \|a\|\|b\|$.
In this case, we call A a normed algebra.
A complete normed algebra is called Banach.
If A possesses a unit 1_A and $\|1_A\| = 1$, we say that A is unital.

So $\mathcal{B}(\mathcal{H})$ is a unital Banach algebra.

Definition

An involution of an algebra A is a conjugate linear map $* : A \to A$ such that $(ab)^* = b^* a^*$ and $a^{**} = a$.

We've seen an involution map on $\mathcal{B}(\mathcal{H})$ via the adjoints.

Definition

A Banach $*$-algebra A is a Banach algebra with an involution such that $\|a^*\| = \|a\|$.

Then $\mathcal{B}(\mathcal{H})$ is also a $*$-algebra.

Definition

A C^*-algebra A is a $*$-algebra such that $\|a^* a\| = \|a\|^2$.

And so we finally see that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Take a set $S \subseteq A$ where A is a C^*-algebra. The C^*-algebra generated by S is nothing more than the intersection of all C^*-subalgebras of A containing this set. We usually write $C^*(S)$.

We denote the C^*-algebra generated by a set $\{T_1, \ldots T_n\} \subseteq \mathcal{B}(\mathcal{H})$,

$$C^*(T_1, \ldots, T_n)$$

For $T_1, \ldots, T_n$ satisfying certain relations, this is the C^*-algebra introduced by Cuntz.

Let's make a few points,

- $\mathcal{B}(\mathcal{H})$ is the case example of a C^*-algebra.
- Not only that, but we may prove that for any C^*-algebra A, there exists a special one to one map $\pi : A \to \mathcal{B}(\mathcal{H})$.
- Every C^*-algebra is in a way a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.
- $\mathcal{B}(\mathcal{H})$ is not in general commutative.
- For $\mathcal{H} = \mathbb{C}$ and dimension n, we get the C^*-algebra $M_n(\mathbb{C})$ as another well-known example.

Finally, we note that,

- We can talk about projections in a C^*-algebra as the elements satisfying $p^2 = p^* = p$.
- There is a way to apply an ordering on a C^*-algebra and talk about positivity. Projections are positive elements.
- With the help of this ordering, we can find a way to also talk about partial isometries in a C^*-algebra.

We only pick and choose related concepts and propositions for this presentation. This is always in the context of $\mathcal{B}(\mathcal{H})$.

Digraph is a fancy word for directed graph. A directed graph

$$E = (E^0, E^1, r, s)$$

consists of two countable sets E^0, E^1 and two maps $r,\ s : E^1 \to E^0$.

- E^0 is the set of vertices
- E^1 is the set of edges
- s is the "source" map, r is the "range" map
- e is the directed path from $s(e)$ to $r(e)$
- $s(e)$ emits e and $r(e)$ receives it

We call E finite if both E^0 and E^1 are finite. For example,

$$\begin{tikzcd} v \arrow[loop left, "e"] & w \arrow[l, "f"'] \end{tikzcd}$$

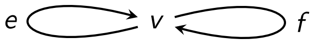
In this case, $E^0 = \{v, w\}$, $E^1 = \{e, f\}$. The maps $r,\ s$ satisfy

$$r(e) = v,\ \ r(f) = v$$

$$s(e) = v,\ \ s(f) = w$$

The edge e is a (directed) loop.

We also allow numerous loops, looping on the same vertex.



And we may also have infinite sets of vertices or edges.

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots$$

If $|r^{-1}(v)| < +\infty$ for all $v \in E^0$, we say that the (directed) graph is row-finite.

There are various definitions and interesting facts, but we only give this basic overview.

The aim is to model, in a way, a directed graph E using elements of $\mathcal{B}(\mathcal{H})$.

- For each $v \in E^0$ we consider a projection P_v
- P_v should be mutually orthogonal
- For each $e \in E^1$ we consider a partial isometry S_e
- The Cuntz-Krieger relations have to hold,

$$\text{(CK1)} \quad S_e^* S_e = P_{s(e)}, \quad \text{for all} \quad e \in E^1$$

$$\text{(CK2)} \quad P_{r(e)} = \sum_{e \in r^{-1}(v)} S_e S_e^*, \quad \text{when} \quad r^{-1}(v) \neq \varnothing$$

The family $\{S, P\}$ is called a Cuntz-Krieger E-family.

Let's make a few points,

- It is apparent that this works well for row-finite graphs. We will only consider row-finite graphs in this presentation.
- The use of s and r may even be interchanged, but the conventions we use is what prevailed.
- There are various things to consider here, but we only state things for row-finite graphs.
- These relations generalize the ones that form the family generating Cuntz's construction.

There are various things of interest here. For example,

- $S_e S_e^*$ are mutually orthogonal projections
- $S_e = P_{r(e)} S_e = S_e P_{s(e)}$
- $P_v \mathcal{H} = \bigoplus_{e \in r^{-1}(v)} S_e \mathcal{H}$

but we won't be considering the details for our purposes, as this could get rather technical.

Let's try this out by modeling a graph! Take,

Take $\mathcal{H} = \ell^2(\mathbb{N})$ the space of square summable sequences. This is the case example for a separable Hilbert space. Consider $\{e_n\}$ its orthonormal base and,

$$P_v = I, \quad S_e(e_n) = e_{2n}, \; S_f(e_n) = e_{2n+1}$$

We then can calculate

$$S_e^*(e_n) = e_{\frac{n-1}{2}}, \text{when } n \text{ is odd}, \qquad S_f^*(e_n) = e_{n/2}, \text{when } n \text{ is even}$$

This is a CK E-family for our two-loop graph, meaning,

$$S_e^* S_e = P_v = I, \qquad S_f^* S_f = P_v = I$$

and

$$S_e S_e^* + S_f S_f^* = P_v = I$$

hold.

We can actually show that for modeling of our two-loop graph with non-zero projections, $\mathcal{H}$ must be infinite dimensional. This is not "a must" for all graphs. We give such an example in the next.

The simplest one-loop graph

may be "modeled" in $\mathcal{H} = \mathbb{C}$. Take $P_v = id_{\mathbb{C}}$ and any unitary operator on $\mathbb{C}$, say $S_e(z) = ze^{i\theta}$.

An interesting note is that, in any modeling in a Hilbert space $\mathcal{H}$, then S_e must be unitary on an closed subspace of $\mathcal{H}$. Indeed, in the case of our graph, the CK relations give that $S_e S_e^* = P_v = S_e^* S_e$.

Ok great, but how does this give us information about the graph? An enlightening proposition is the following,

Proposition

Suppose E *is a row-finite graph and* $\{S, P\}$ *is a CK family on* B. *Then for all* $e, f \in E^1$,

- $S_e S_f \neq 0 \implies s(e) = r(f)$: *there is a path from* f *to* e
- $S_e S_f^* \neq 0 \implies s(e) = s(f)$: *both* e *and* f *start on the same vertex*

Using our modeling, we may also find out, for example,

- If a vertex is a source
- If we have sinks
- If E has loops

Unfortunately these use rather advanced tools for the scope of this presentation.

We only present, without getting into any of the details, the graph algebra of E, denoted by $C^*(E)$.

Theorem (The Universal Graph Algebra $C^*(E)$)

Take a row-finite graph E. Then there exists a C^*-algebra B and a non-zero CK family $\{S, P\}$ such that:

- $B = C^*(S, P)$
- *If $\{S', P'\}$ is another CK family on a C^*-algebra C generating C, then there exists a $*$-epimorphism $\Phi : B \to C$ such that $\Phi(S_e) = S'_e$ and $\Phi(P_v) = P'_v$ for all $v \in E^0$ and $e \in E^1$.*

The second property is called the universal property for a C^*-algebra.

Some final comments on this,

- The universal algebra $C^*(E)$ is unique up to a $*$-isomorphism
- This is not the only algebra associated with a digraph E. It can actually be seen as a quotient of another, more "general" C^*-algebra related to a digraph E.

Cuntz, in 1977, introduced the C^*-algebra generated by a family of $n \geq 2$ isometries on an infinite dimensional Hilbert space $\mathcal{H}$.

Specifically, let $S_1, \ldots, S_n$ be n-isometries such that,

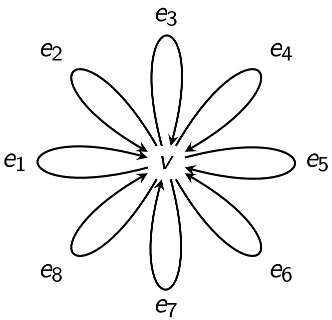
$$S_1 S_1^* + \cdots + S_n S_n^* = I$$

We've seen such an example for $n = 2$! We also noted that the dimension must be infinite.

We can actually show that, for many constructions, we may take a separable infinite dimensional Hilbert space and decompose it accordingly to create a CK E-family!

Of course, we may interpret this construction,

- I is the projection corresponding to a single vertex v
- $S_1, \ldots S_n$ correspond to n edges $e_1, \ldots, e_n$



Cuntz's work was very influential for much of the analysis that followed in related topics and more explicitly, graph algebras.

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Thank you for your attention!