# Jacobi Fields

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# CHAPTER 1

# The Jacobi Equation

## 1.1 The Jacobi Equation

**Definition 1.1** Let $(M, g)$ be a Riemannian manifold and let $\gamma\colon I \to M$ be a geodesic. A variation $\Gamma\colon K \times I \to M$ of $\gamma$ will be called a **variation through geodesics** if each main curve $\Gamma_s$ is a geodesic.

**Example 1.1** We refer the reader to the book [4] and to the proof of Gauss's Lemma, where the variation defined in the corresponding proof is a variation through geodesics.

**Motivation 1.1**

- We will attempt to derive an equation that must be satisfied by the variation field of a variation through geodesics. Nevertheless, we shall see that the resulting equation, given a geodesic $\gamma$, characterizes a class of vector fields along $\gamma$, the so-called **Jacobi fields**, with which we shall be concerned to a large extent throughout the chapters.

- Under the hypotheses of Definition 1.1, let $V(t) = \partial_s \Gamma(0, t)$ be the corresponding variation field of $\Gamma$. Since $\Gamma_s$ is a geodesic for each $s \in J$, it follows that $D_t T = 0$. Using this last relation, we aim to derive a relation involving $V$.

- If we knew that for every vector field $W$ along $\Gamma$ it holds that

  $$D_s D_t W = D_t D_s W,$$

  then by a well-known lemma we would obtain

  $$0 = D_s D_t T = D_t D_s T = D_t D_t S,$$

  and evaluating at $(0, t)$ we would have $D_t^2 V = 0$. However, the truth is quite far from this.

- In the present case, the role of the Riemann curvature tensor is decisive, as it provides a "measure" of how far the difference

  $$D_s D_t W - D_t D_s W$$

  is from being zero.

**Lemma 1.1** Let $(M, g)$ be a Riemannian manifold and let $\Gamma\colon J \times I \to M$ be a smooth one-parameter family of curves. If $V$ is a $\mathscr{C}^\infty$ vector field along $\Gamma$, then

$$D_s D_t V - D_t D_s V = \mathrm{R}(S, T)V,$$

where $S(s,t) = \partial_s \Gamma(s,t)$ and $T(s,t) = \partial_t \Gamma(s,t)$.

*Proof.* It suffices to prove the desired identity in a local coordinate system. Let $(U, (x^i))$ be a coordinate chart on $M$. Then, in coordinates,

$$\Gamma = \left(\Gamma^1, \ldots, \Gamma^n\right),$$

and

$$S(s,t) = \frac{\partial \Gamma^i}{\partial s}(s,t)\,\partial_i\big|_{\Gamma(s,t)}, \quad T(s,t) = \frac{\partial \Gamma^j}{\partial t}(s,t)\,\partial_j\big|_{\Gamma(s,t)}, \quad V(s,t) = V^k(s,t)\,\partial_k\big|_{\Gamma(s,t)}.$$

We have

$$D_t V = D_t\left(V^i \partial_i\right) = \frac{\partial V^i}{\partial t}\partial_i + V^i D_t(\partial_i),$$

and therefore

$$D_s D_t V = \frac{\partial^2 V^i}{\partial s \partial t}\partial_i + \frac{\partial V^i}{\partial t} D_s(\partial_i) + \frac{\partial V^i}{\partial s} D_t(\partial_i) + V^i D_s D_t(\partial_i). \tag{1.1}$$

By symmetry we have

$$D_t D_s V = \frac{\partial^2 V^i}{\partial t \partial s}\partial_i + \frac{\partial V^i}{\partial s} D_t(\partial_i) + \frac{\partial V^i}{\partial t} D_s(\partial_i) + V^i D_t D_s(\partial_i). \tag{1.2}$$

Subtracting (1.2) from (1.1), we obtain

$$D_s D_t V - D_t D_s V = V^i\left[D_s D_t(\partial_i) - D_t D_s(\partial_i)\right].$$

By extendability, we have

$$D_t(\partial_i) = \nabla_T \partial_i = \frac{\partial \Gamma^j}{\partial t}\nabla_{\partial_j}\partial_i.$$

Since $\nabla_{\partial_j}\partial_i$ is also extendable, it follows that

$$D_s D_t \partial_i = \frac{\partial^2 \Gamma^j}{\partial s \partial t}\nabla_{\partial_j}\partial_i + \frac{\partial \Gamma^j}{\partial t}\frac{\partial \Gamma^k}{\partial s}\nabla_{\partial_k}\left(\nabla_{\partial_j}\partial_i\right).$$

By symmetry with respect to $s, t$ and the symmetry of the connection, we obtain

$$D_s D_t \partial_i - D_t D_s \partial_i = \frac{\partial \Gamma^j}{\partial t}\frac{\partial \Gamma^k}{\partial s}\left[\nabla_{\partial_k}\left(\nabla_{\partial_j}\partial_i\right) - \nabla_{\partial_j}\left(\nabla_{\partial_k}\partial_i\right)\right] = \mathrm{R}(S,T)\partial_i,$$

where R denotes the Riemann curvature tensor, and the last equality follows from the $\mathscr{C}^\infty$-linearity of R. The desired result follows immediately from the above relation. □

**Theorem 1.1** (Jacobi Equation)**.** *Let $(M, g)$ be a Riemannian manifold, let $\gamma\colon I \to M$ be a geodesic, and let $V \in \mathscr{X}(\gamma)$. Suppose that $V$ is the variation field of some* *variation of $\gamma$ through geodesics. Then*

$$D_t^2 V + R(V, \dot{\gamma})(\dot{\gamma}) = 0. \tag{1.3}$$

*Proof.* Let $\Gamma \colon J \times I \to M$ be a variation of $\gamma$ through geodesics such that

$$V(t) = \partial_s \Gamma(0, t).$$

Since $\Gamma_s$ is a geodesic for each $s \in J$, we have $D_t T = 0$, and hence

$$0 = D_s D_t T = D_t D_s T = D_t D_t S + \mathrm{R}(S, T)T,$$

where the second equality follows from the previous lemma and the third from the Symmetry Lemma. Applying the last relation at $(0, t)$, and noting that

$$T(0, t) = \dot{\gamma}(t) \quad \text{and} \quad S(0, t) = \partial_s \Gamma(0, t) = V(t),$$

we obtain the desired relation. □

**Definition 1.2** Let $(M, g)$ be a Riemannian manifold and let $\gamma$ be a geodesic. A vector field $V \in \mathscr{X}(\gamma)$ will be called a **Jacobi field** if it satisfies equation 1.3.

**Remark 1.1**

- Let $(M, g)$ be a Riemannian manifold, let $\gamma \colon I \to M$ be a geodesic of $M$, and let $J \in \mathscr{X}(\gamma)$ be a Jacobi field. Let $p = \gamma(a)$ with $a \in I$.
- In the case of the equations for parallel vector fields along a curve and of the geodesic equations, it was quite convenient to study these equations using a local coordinate system. This was a decisive step in order to prove existence and uniqueness theorems.
- Due to the appearance of a second-order covariant derivative in the Jacobi equation, it is more convenient, for a Jacobi field $J \in \mathscr{X}(\gamma)$, to consider a parallel orthonormal frame $\{E_i\}$ along $\gamma$. Then
$$J(t) = J^i(t) E_i(t),$$
and equation 1.3 can be rewritten in the following form:
$$\ddot{J}^i(t) + R^i_{jk\ell} \circ \gamma(t)\, J^j(t) J^k(t) J^\ell(t) = 0, \tag{1.4}$$
where
$$\mathrm{R}(E_j, E_k)E_\ell = R^i_{jk\ell} E_i.$$
- Using the above equation and given two initial conditions, we will show that, for a given geodesic, there exists a unique Jacobi field satisfying these initial conditions.

**Theorem 1.2** (Existence and Uniqueness of Jacobi Fields)**.** *Let $(M, g)$ be a Riemannian manifold, let $\gamma \colon I \to M$ be a geodesic of $M$, and let $a \in I$. If $v, w \in T_p M$, where $p = \gamma(a)$, then there exists a unique Jacobi field $J \colon I \to TM$ such that*

$$J(a) = v \quad \textit{and} \quad D_t J(a) = w.$$

*Proof.* Let $\{E_i\}_i$ be a parallel orthonormal frame along $\gamma$. Then, by the previous remark, we are reduced to solving the second-order system of differential equations

$$\ddot{J}^i(t) + R^i_{jk\ell} \circ \gamma(t)\, J^j(t) J^k(t) J^\ell(t) = 0.$$

Setting $W^i = \dot{J}^i$, we obtain a linear first-order system of differential equations, this time with $2n$ unknowns:

$$W^i = \dot{J}^i,$$
$$\dot{W}^i = -\left(R^i_{jk\ell} \circ \gamma\right) J^j J^k J^\ell.$$

If $v = v^i E_i(a)$ and $w = w^i E_i(a)$, then the initial conditions of the above initial value problem are

$$\left(J^1(a), \ldots, J^n(a), W^1(a), \ldots, W^n(a)\right) = (v_1, \ldots, v_n, w_1, \ldots, w_n).$$

By a well-known theorem, the above system admits a unique solution. Moreover, due to the above initial conditions, the field $J = J^i E_i$ satisfies the Jacobi equation and

$$J(a) = J^i(a) E_i(a) = v^i E_i(a) = v, \quad \text{and} \quad D_t J(a) = \dot{J}^i(a) E_i(a) = w.$$

□

**Remark 1.2**
- Let $(M, g)$ be a Riemannian manifold, let $\gamma\colon I \to M$ be a geodesic of $M$, and let $a \in I$. If $p = \gamma(a)$, then by the previous theorem we have shown that there exists a bijective mapping

  $$\Phi\colon \mathscr{J}(\gamma) \to T_pM \oplus T_pM, \quad \Phi(J) = (J(a), D_t J(a)),$$

  where $\mathscr{J}(\gamma)$ denotes the set of Jacobi fields along $\gamma$.

- Moreover, it is easy to verify, due to equation 1.3, that $\mathscr{J}(\gamma)$ is a vector subspace of $\mathscr{X}(\gamma)$.

- Owing to the linearity of the Jacobi equation, it follows that $\Phi$ is a linear map and hence a linear isomorphism.

**Corollary 1.1** Let $(M, g)$ be of dimension $n$ and let $\gamma\colon I \to M$ be a geodesic of $M$. Then $\mathscr{J}(\gamma)$ is a vector subspace of $\mathscr{X}(\gamma)$ of dimension $2n$.

**Motivation 1.2**
- By Theorem 1.1 we showed that, for a given geodesic and a variation of it through geodesics, the corresponding variation field is a Jacobi field.

- Could we claim that the converse also holds? That is, is every Jacobi field the variation field of some variation through geodesics? The following proposition gives an affirmative answer to this question, under certain assumptions.

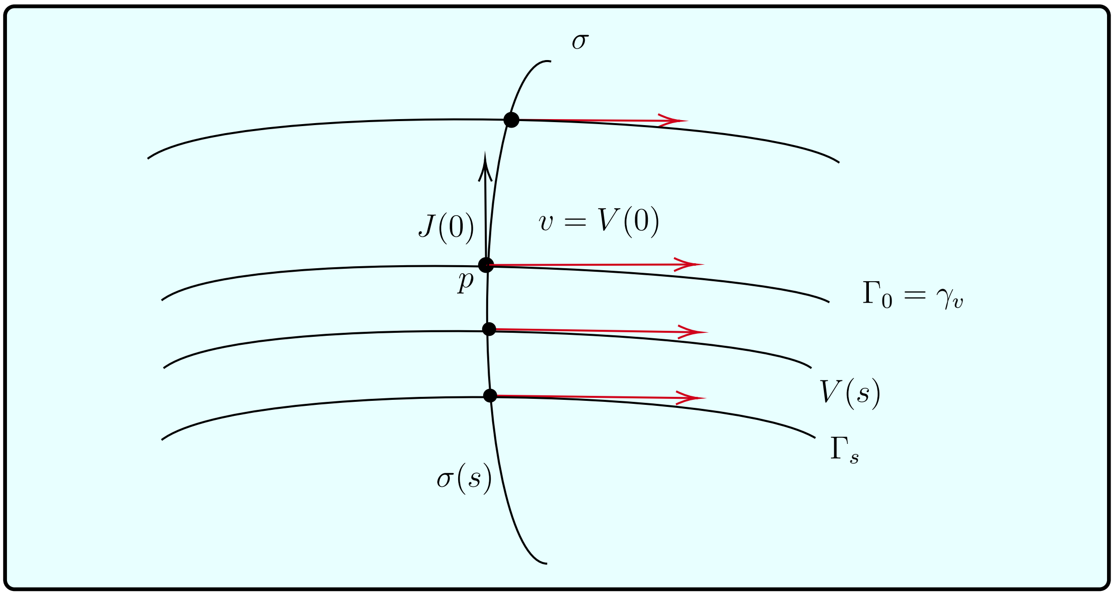
**Proposition 1.1.** *Let $(M, g)$ be a Riemannian manifold and let $\gamma\colon I \to M$ be a geodesic. If $M$ is complete or $I$ is a compact interval, then every Jacobi field along $\gamma$ is the variation field of some variation of $\gamma$ through geodesics.*

*Proof.*

- Without loss of generality, we may assume that $0 \in I$ (by applying an appropriate translation in the parameter $t$). We then denote $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, that is,

$$\gamma(t) = \exp_p(tv).$$

- Let $J \in \mathscr{J}(\gamma)$ and consider a curve $\sigma\colon (-\varepsilon, \varepsilon) \to M$ such that $\sigma(0) = p$ and $\dot{\sigma}(0) = J(0)$. In addition, choose a vector field $V(s) \in \mathscr{X}(\sigma)$ with $V(0) = v$ and $D_s V(0) = D_t J(0)$.




- We define $\Gamma(s,t) = \exp_{\sigma(s)}(tV(s))$. Using the fact that either $M$ is complete, by the Hopf–Rinow theorem, or that $I$ is a compact interval, we may conclude that

$$\Gamma\colon (-\delta, \delta) \times I \to M$$

is a variation of $\gamma$. It is left as an exercise to the reader to show that $\Gamma$ is a variation of $\gamma$ through geodesics.

- We consider the variation field $W(t)$ of $\Gamma$. We observe that

$$W(0) = \dot{\sigma}(0) = J(0), \qquad D_t W = D_t S(0,0) = D_s T(0,0) = D_s V(0) = D_t J(0).$$

By Theorem 1.1, the field $W$ is a Jacobi field, and from the above relations together with the uniqueness of Jacobi fields, it follows that $W = J$.

□

**Motivation 1.3** Let $(M, g)$ be a Riemannian manifold and let $\gamma\colon I \to M$ be a $\mathscr{C}^\infty$ curve. Due to the natural behavior of the Levi–Civita connection under local isometries, we have already seen the following result: if

$$F\colon (M, g) \to (\tilde{M}, \tilde{g})$$

is a local isometry and $\tilde{\gamma} = F \circ \gamma$, and if $\gamma$ is a geodesic of $M$, then $\tilde{\gamma}$ is a geodesic. Can the previous result yield an analogous statement for two vector fields $J \in \mathscr{X}(\gamma)$ and $\tilde{J} \in \mathscr{X}(\tilde{\gamma})$ when these are $F$–related?

**Proposition 1.2.** *Let $F\colon (M,g) \to \left(\tilde{M},\tilde{g}\right)$ be a local isometry between two Riemannian manifolds and let $\gamma\colon I \to M$ be a geodesic (hence $\tilde{\gamma} = F \circ \gamma$ is also a geodesic). If two vector fields $J \in \mathscr{X}(\gamma)$ and $\tilde{J} \in \mathscr{X}(\tilde{\gamma})$ are $F$-related, that is,*

$$d_{\gamma(t)}F(J(t)) = \tilde{J}(t),$$

*then $J \in \mathscr{J}(\gamma)$ if and only if $\tilde{J} \in \mathscr{J}(\tilde{\gamma})$.*

*Proof.* The result follows locally by using the naturality of the covariant derivative and the naturality of the Riemann curvature operator. □

## 1.2 Tangential and Normal Jacobi Fields

**Motivation 1.4**

- It is natural, when one begins studying Jacobi fields, to look for trivial examples and to examine what information one can extract from them.
- If $(M,g)$ is a Riemannian manifold and $\gamma\colon I \to M$ is a geodesic, then by the $\mathscr{C}^\infty$-linearity as well as the symmetries of the curvature tensor it follows that

  $$J_0(t) = \dot{\gamma}(t) \quad \text{and} \quad J_1(t) = t\dot{\gamma}(t)$$

  are Jacobi fields along $\gamma$.
- If we assume that $M$ is complete or that $I$ is compact, and we consider the corresponding variations from the proof of Proposition 1.1 whose variation fields are $J_0$, $J_1$ respectively, we observe that

  $$\Gamma_0(s,t) = \gamma(s+t) \quad \text{and} \quad \Gamma_1(s,t) = \gamma((1+s)t).$$

  We therefore observe that the information we obtain from the above relations is negligible with respect to the behavior of geodesics other than $\gamma$.
- If we consider an arbitrary regular curve $\gamma\colon I \to M$ in a Riemannian manifold $(M,g)$, then it is an immersion and for each $t \in I$ we have

  $$T^\top_{\gamma(t)}M := \langle\dot{\gamma}\rangle \le T_{\gamma(t)}M,$$

  which is a 1-dimensional subspace; hence we may consider $T^\perp_{\gamma(t)}M$ to be the corresponding $(n-1)$-dimensional subspace orthogonal to $T^\top_{\gamma(t)}M$.
- A vector field $V \in \mathscr{X}(\gamma)$ will be called **tangential** if $V(t) \in T^\top_{\gamma t}M$ for every $t \in I$. A vector field $V \in \mathscr{X}(\gamma)$ will be called **normal** if $V(t) \in T^\perp_{\gamma(t)}M$ for every $t \in I$. The spaces of tangential and normal vector fields along $\gamma$ will be denoted by $\mathscr{X}^\top(\gamma)$ and $\mathscr{X}^\perp(\gamma)$, respectively.

**Definition 1.3** Let $(M,g)$ be a Riemannian manifold and let $\gamma\colon I \to M$ be a geodesic. A vector field $V \in \mathscr{X}(\gamma)$ will be called a **tangential Jacobi field** if it is a Jacobi field and $V(t) \in T^\top_{\gamma t}M$ for every $t \in I$. The normal Jacobi fields are defined analogously. The spaces of tangential and normal Jacobi fields along $\gamma$ will be denoted by $\mathscr{J}^\top(\gamma)$ and $\mathscr{J}^\perp(\gamma)$, respectively.

**Proposition 1.3.** *Let $(M, g)$ be a Riemannian manifold, let $\gamma\colon I \to M$ be a geodesic, and let $J \in \mathscr{J}(\gamma)$. The following statements are equivalent.*

*(a) $J$ is a normal Jacobi field along $\gamma$.*

*(b) $J$ is orthogonal to $\dot\gamma$ at two distinct points.*

*(c) $J$ and $D_t J$ are orthogonal to $\dot\gamma$ at one point.*

*(d) $J$ and $D_t J$ are orthogonal to $\dot\gamma$ at every point.*

*Proof.* Let $f\colon I \to \mathbb{R}$ be defined by

$$f(t) = \langle J(t), \dot\gamma(t)\rangle .$$

Since the Levi–Civita connection is metric, we have

$$\dot f = \langle D_t J, \dot\gamma\rangle + \langle J, D_t\dot\gamma\rangle = \langle D_t J, \dot\gamma\rangle ,$$

and similarly

$$\ddot f = \left\langle D_t^2 J, \dot\gamma\right\rangle = -\left\langle \mathrm{R}(J, \dot\gamma)\dot\gamma, \dot\gamma\right\rangle = -R_m(J, \dot\gamma, \dot\gamma, \dot\gamma) = 0,$$

where the last equality follows from the symmetries of the Riemann curvature tensor. Consequently, $\dot f$ is constant, from which the equivalences (a)–(d) follow immediately. □

**Corollary 1.2** Let $(M, g)$ be a Riemannian manifold and let $\gamma\colon I \to M$ be a nonconstant (that is, regular) geodesic. Then $\mathscr{J}^\perp(\gamma)$ is a $(2n-2)$-dimensional subspace of $\mathscr{J}(\gamma)$ and $\mathscr{J}^\top(\gamma)$ is a 2-dimensional subspace of $\mathscr{J}(\gamma)$. Consequently,

$$\mathscr{J}(\gamma) = \mathscr{J}^\top(\gamma) \oplus \mathscr{J}^\perp(\gamma).$$

*Proof.*

- We consider the map

  $$\Phi\colon \mathscr{J}(\gamma) \to T_pM \oplus T_pM, \quad \Phi(J) = (J(a), D_t J(a)),$$

  which we have shown to be a linear isomorphism. From the previous proposition it is clear that

  $$\Phi\big(\mathscr{J}^\perp(\gamma)\big) = T_pM^\perp \oplus T_pM^\perp,$$

  where the latter space has dimension $2n-2$.

- It is immediate that

  $$\mathscr{J}^\top(\gamma) \cap \mathscr{J}^\perp(\gamma) = \{0\}.$$

  Moreover, since the Jacobi fields $J_0, J_1$ defined in Motivation 1.4 belong to $\mathscr{J}^\top(\gamma)$ and are linearly independent, it follows that

  $$\dim \mathscr{J}^\top(\gamma) = 2.$$

□

CHAPTER 2

# Special Types of Jacobi Fields

## 2.1 Jacobi Fields Vanishing at a Point

**Lemma 2.1** Let $(M, g)$ be a Riemannian manifold, let $I \subseteq \mathbb{R}$ be an interval containing $0$, let $\gamma\colon I \to M$ be a geodesic, and let $J \in \mathscr{J}(\gamma)$ such that $J(0) = 0$. Assume that $M$ is complete or that $I$ is compact. Then $J$ is the variation field of the variation of $\gamma$

$$\Gamma(s,t) = \exp_p(t(v + sw)),$$

where $\gamma(0) = p$, $\dot\gamma(0) = v$, and $D_t J(0) = w$.

*Proof.* Following the steps of the proof of Proposition 1.1, we choose $\sigma \equiv p$ and $W(s) = v + sw$. Then the desired variation is

$$\Gamma(s,t) = \exp_{\sigma(s)}(tW(s)) = \exp_p(t(v + sw)).$$

□

**Proposition 2.1.** *Let $(M, g)$ be a Riemannian manifold, let $I \subseteq \mathbb{R}$ be an interval containing $0$, and let $\gamma\colon I \to M$ be a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. For every $w \in T_pM$, the Jacobi field $J \in \mathscr{J}(\gamma)$ satisfying*

$$J(0) = 0 \quad \textit{and} \quad D_t J(0) = w$$

*is given by the formula*

$$J(t) = d_{tv}(\exp_p)(tw), \tag{2.1}$$

*under the identification $T_{tv}(T_pM) \equiv T_pM$.*

*Proof.* Since each $t \in I$ belongs to some compact interval $0 \in I_0 \subseteq I$, it follows that for every such compact interval $I_0$, the Jacobi field $J$ is, by Lemma 2.1, the variation field of the variation

$$\Gamma(s,t) = \exp_p(t(v + sw)).$$

We compute as follows:

$$\begin{aligned} J(t) &= \partial_s\Gamma(0,t) = d_{(0,t)}\left(\exp_p(t(v + sw))\right)(\partial_s) \\ &= d_{tv}(\exp_p) \circ d_{(0,t)}(t(v + sw))(\partial_s) = d_{tv}(\exp_p)(tw). \end{aligned}$$

□

**Remark 2.1** Under the hypotheses of the previous proposition, assume that $(U,(x^i))$ is a normal neighborhood with normal coordinates around $p$ such that $\gamma(I) \subseteq U$, with

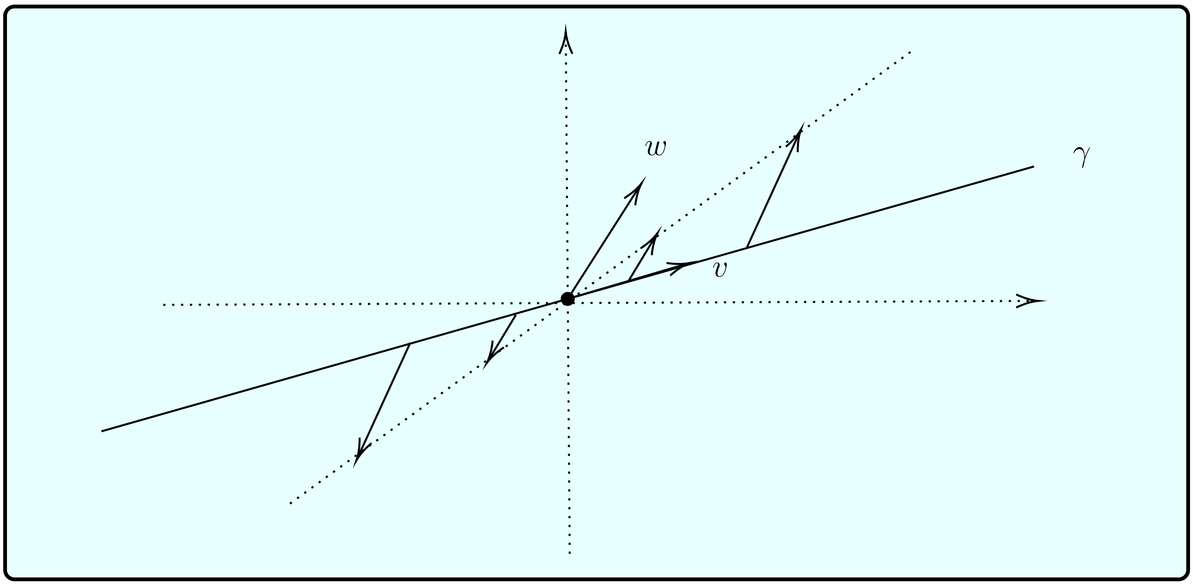
$$v = v^i \partial_i\big|_p \quad \text{and} \quad w = w^i \partial_i\big|_p.$$

Then the corresponding representation of $\Gamma$ is

$$\Gamma(s,t) = \left(t\left(v^1 + sw^1\right), \ldots, t\left(v^n + sw^n\right)\right),$$

and since $J$ is the variation field of $\Gamma$, it can be written in normal coordinates in the form

$$J(t) = tw^i \partial_i\big|_{\gamma(t)}. \tag{2.2}$$




**Remark 2.2** Let $(M,g)$ be a Riemannian manifold, let $p \in M$, and let $U$ be a normal neighborhood of $p$. Let $q \in U \setminus \{p\}$. Consider the radial geodesic

$$\gamma(t) = \exp_p(tv),$$

where $v = \exp_p^{-1}(q)$. Then $\gamma(1) = q$. Let $w \in T_qM$, which in normal coordinates can be written as

$$w = w^i \partial_i\big|_{\gamma(t)}.$$

Setting $v' = w^i \partial_i\big|_p$ and applying the previous remark, we obtain that the Jacobi field $J \in \mathscr{J}(\gamma)$ vanishing at 0 satisfies

$$J(1) = w^i \partial_i\big|_{\gamma(1)} = w^i \partial_i\big|_q = w.$$

**Corollary 2.1** Let $(M,g)$ be a Riemannian manifold, let $p \in M$, and let $U$ be a normal neighborhood of $p$. Let $q \in U \setminus \{p\}$. Consider the radial geodesic

$$\gamma(t) = \exp_p(tv),$$

where $v = \exp_p^{-1}(q)$. Then $\gamma(1) = q$. For every $w \in T_qM$, there exists a Jacobi field $J \in \mathscr{J}(\gamma)$ that vanishes at $t = 0$ and satisfies $J(1) = w$.

## 2.2 Jacobi Fields in Spaces of Constant Curvature

**Motivation 2.1** On a Riemannian manifold $(M, g)$, for any linearly independent $v, w \in T_pM$ and $\Pi$ the plane spanned by $v, w$, we can define the **sectional curvature** of $\Pi$ by

$$K(\Pi) = \frac{\mathrm{Rm}_p(v, w, w, v)}{|u \wedge w|^2},$$

where

$$|u \wedge w| = \left(\langle v, v\rangle \cdot \langle w, w\rangle - \langle v, w\rangle\right)^{1/2}.$$

Spaces of constant sectional curvature are quite interesting, although this may not be apparent at first glance, and their properties come into direct contact with Jacobi fields. To give the reader a first taste, we recall that every connected Riemannian manifold admits a universal covering (see [3]), whose nature as a manifold is generally unknown. In the special case when $M$ has constant sectional curvature $-1, 0, 1$, there are not many possibilities for this covering. It must be one of the following:

$$\mathbb{H}^n, \quad \mathbb{R}^n, \quad \mathbb{S}^n.$$

**Lemma 2.2** Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $c$. Then for all $v, w, x \in T_pM$ we have

$$\mathrm{Rm}(v, w)(x) = c\left(\langle w, x\rangle v + \langle v, x\rangle w\right),$$

where Rm denotes the map $\mathrm{Rm}\colon T_pM \times T_pM \times T_pM \to T_pM$ induced by the Riemann curvature tensor.

**Definition 2.1** Let $c \in \mathbb{R}$. We denote by $s_c\colon \mathbb{R} \to \mathbb{R}$ the function

$$s_c(t) = \begin{cases} t, & c = 0, \\ R\sin(t/R), & c = 1/R^2 > 0, \\ R\sinh(t/R), & c = -1/R^2 < 0. \end{cases}$$

**Proposition 2.2.** *Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $c$ and let $\gamma\colon I \to M$ be a unit-speed geodesic of $M$. If $J \in \mathscr{J}(\gamma)^\perp$ with $J(0) = 0$, then $J$ is of the form*

$$J(t) = ks_c(t)E(t),$$

*where $s_c$ is the function defined above and $E(t)$ is a parallel, normal, unit vector field along $\gamma$.*

*Proof.*
- Let $E(t)$ be a parallel, normal, unit vector field along $\gamma$ and let $J \in \mathscr{J}(\gamma)^\perp$ with $J(0) = 0$, written in the form

  $$J(t) = u(t)E(t).$$

  Then we have

  $$D_t^2 J + R(J, \dot\gamma)(\dot\gamma) = \ddot u + cu = 0,$$

  where the equality follows immediately from the previous lemma, from the fact that $J$ is normal, and from the assumption that $\dot\gamma$ has unit speed.

- The last equation is an ordinary second-order differential equation with constant coefficients, and the solutions satisfying the condition $u(0) = 0$ are precisely the constant multiples $ks_c$.

- By a dimension-counting argument, it follows that the normal Jacobi fields vanishing at $t = 0$ are exactly those of the form

$$J(t) = ks_c(t)E(t).$$

□

**Remark 2.3** Consider the map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ defined by $\pi(x) = x/\|x\|$. If $g^\circ \in \mathscr{T}^{(0,2)}(M)$ denotes the Riemannian metric induced on the sphere from the standard Euclidean metric on $\mathbb{R}^n$, then we consider the pullback tensor

$$\hat{g} = \pi^* g^\circ \in \mathscr{T}^{0,2}\left(\mathbb{R}^{n+1} \setminus \{0\}\right).$$

How can we relate the standard Euclidean metric to the tensor $\hat{g}$ above?

**Lemma 2.3** On $\mathbb{R}^n \setminus \{0\}$, the standard Euclidean metric $\overline{g}$ can be written as

$$\overline{g} = dr \otimes dr + r^2 \hat{g},$$

where $r$ is the standard Euclidean distance function $r(x) = \|x\|$.

*Proof.* Consider the map $\Phi\colon \mathbb{R}_+ \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ defined by $\Phi(\rho, x) = \rho x$. Equipping $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ with the warped product metric $d\rho \otimes d\rho + \rho^2 g^\circ$, we obtain

$$\overline{g} = \left(\Phi^{-1}\right)\left(d\rho \otimes d\rho + \rho^2 g^\circ\right) = dr \otimes dr + r^2 \hat{g}.$$

□

The following theorem, which we state, can be found in [4].

**Theorem 2.1.** *Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $c$. Let $p \in M$ and let $(U, (x^i))$ be normal coordinates around $p$. If $r$ denotes the radial distance defined on $U \setminus \{p\}$ and $\hat{g} \in \mathscr{T}^{0,2}(U \setminus \{p\})$ is defined in $x$–coordinates as before, then*

$$g = dr \otimes dr + s_c(r)^2 \hat{g}.$$

*Proof.*
- For practical reasons, we denote by $\overline{g}$ the Euclidean inner product (in normal coordinates) and by $g_c$ the right-hand side of the above equality. Let $q \in U \setminus \{p\}$. If $b = r(q)$, then by Gauss' Lemma, every $v \in T_qM$ admits an (orthogonal) decomposition of the form

$$v = V^\perp + V^\top,$$

where $V^\perp$ is a multiple of $\partial_r|_q$ and $V^\top$ is tangent to the geodesic sphere of radius $b$.

- We wish to show that $g(v, v) = g_c(v, v)$ (and then apply this to $v + w$). By the properties of normal coordinates, $\partial_r$ is unit with respect to $g$, $\overline{g}$, and $g_c$. Therefore, it suffices to prove the equality assuming that $v$ is tangent to the geodesic sphere of radius $b$.

- For such vectors $v$, since $r \equiv b$, it suffices to show that

$$g(v, v) = s_c(b)^2 \hat{g}(b).$$

From Lemma 2.3 we have

$$g_c(v, v) = \frac{s_c(b)^2}{b^2} \overline{g}(v, v).$$

- By Lemma 2.2, we may assume that for the radial geodesic $\gamma\colon [0, b] \to M$ (in normal coordinates)

$$\gamma(t) = \left( \frac{t}{b} q^1, \ldots, \frac{t}{b} q^n \right),$$

there exists a Jacobi field $J \in \mathscr{J}(\gamma)$ such that $J(b) = v$, given by

$$J(t) = \frac{t}{b} v^i \partial_i \big|_{\gamma(t)}.$$

- Since $J(0) = 0$ and $J(b) = v$, and $v$ is orthogonal to $\dot{\gamma}$, it follows that $J \in \mathscr{J}^\perp(\gamma)$, hence it is of the form

$$J(t) = k s_c(t) E(t).$$

- Using the last relation, we obtain the desired equality.

□

**Corollary 2.2** Let $(M, g)$ and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds of the same dimension and constant sectional curvature $c$. Then $(M, g)$ and $(\tilde{M}, \tilde{g})$ are locally isometric.

CHAPTER 3

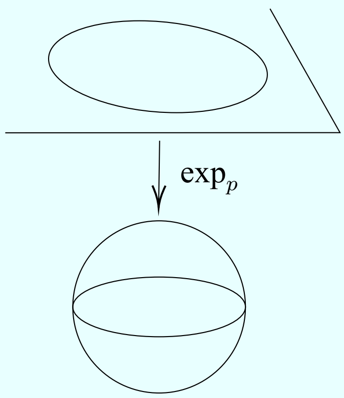
# Conjugate Points

## 3.1 Examples of Conjugate Points

In this chapter we begin to address the question of when, and under what conditions, the exponential map becomes a diffeomorphism, and in particular we will focus on its critical points. After developing the appropriate tools, we will return to this topic again in Chapter 5. We already know that the exponential map $\exp_p$ on the open set

$$\mathscr{E}_p = \{v \in T_pM \mid \exists \gamma : I \supseteq [0,1] \to M \text{ a maximal geodesic with } \gamma(0) = p,\ \dot{\gamma}(0) = v\}$$

is a smooth map between $n$-dimensional spaces, and therefore it admits a local inverse at points where $d_v(\exp_p)$ has rank $n$ (by the inverse function theorem or the rank theorem).




We will call the points of the exponential map at which the inverse function theorem and the rank theorem apply **regular points**. We already know that 0 is a regular point, since

$$d_0(\exp_p) = \text{id}.$$

To gain some intuition about the nature of the critical points of the exponential map, we consider the example of the sphere $\mathbb{S}^2$. There, the exponential map $\exp_p$ is a diffeomorphism when restricted to the ball $B_\pi(0) \subseteq T_p\mathbb{S}^2$. However, every point on the boundary $\partial B_\pi(0)$ is a critical point, which suggests that the exponential map cannot be extended as a diffeomorphism to the antipodal points.

What we aim to show is that Jacobi fields provide a powerful tool for studying these critical points.

**Motivation 3.1** We have already seen, from Proposition 2.2, that every Jacobi field on the sphere $\mathbb{S}^2$ which vanishes at $p$ has its first zero at distance exactly $\pi$ from $p$, namely at the antipodal point. On the other hand, if $U$ is a normal neighborhood of $p$, then Proposition 2.1, in normal coordinates, yields the relation

$$J(t) = tw^i \partial_i|_{\gamma(t)} \quad (w \neq 0),$$

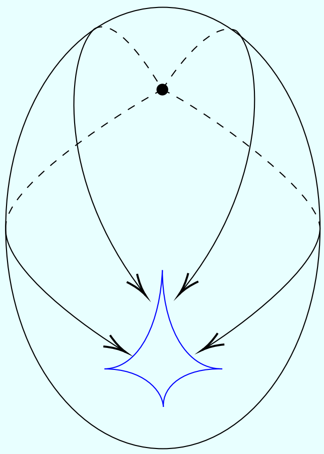
and consequently the Jacobi field does not vanish at any other point within the normal neighborhood.

**Definition 3.1 (Conjugate points).** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a geodesic with $\gamma(a) = p$ and $\gamma(b) = q$. We say that $p$ and $q$ are **conjugate** along $\gamma$ if there exists a Jacobi field

$$J \in \mathscr{J}(\gamma) \setminus \{0\}$$

such that $J(a) = J(b) = 0$. The **multiplicity** (or order) of the conjugacy is the dimension of the space of such Jacobi fields.

The study of conjugate points is far from trivial. The example of the sphere given above may be misleading in this respect. On ellipsoids $\mathcal{E}$ the situation is more complicated but more typical, and the set of first conjugate points from a given point $p$ is a closed curve (depicted in the figure below). More details concerning conjugate points can be found in [1].



## 3.2 Basic Results on Conjugate Points

**Remark 3.1**

- By the existence and uniqueness theorem for Jacobi fields, the space of Jacobi fields vanishing at $a$ has dimension $n$. Since tangential Jacobi fields can vanish at at most one point, the multiplicity of conjugacy must be at most $n - 1$.
- In fact, the inequality is sharp: by Proposition 2.2, on the spheres $\mathbb{S}^n$, for every geodesic joining antipodal points and for every parallel vector field that is normal along $\gamma$, there exists a Jacobi field that vanishes at the endpoints. However, the space of parallel normal vector fields has dimension $n - 1$.

The following proposition will justify our previous intuition by showing that conjugate points are very closely related to the critical points of the exponential map.

**Proposition 3.1** (Critical points of the exponential map)**.** *Let $(M, g)$ be a Riemannian manifold, let $p \in M$, and let $v \in T_pM$. Let $\gamma : [0, 1] \to M$ be the segment of the* *maximal geodesic determined by $v$ (starting at $p$) such that $\gamma(t) = \exp_p(tv)$, and let $q = \exp_p v$. Then $v$ is a critical point of $\exp_p$ if and only if $p$ and $q$ are conjugate along $\gamma$.*

*Proof.* ($\Rightarrow$) If $v$ is a critical point, then there exists a nonzero $w \in T_v T_p M \simeq T_p M$ such that $d_v(\exp_p)(w) = 0$. Consider the variation

$$\Gamma(s,t) = \exp_p\big(t(v+sw)\big)$$

(from Lemma 2.1), as well as the Jacobi field $J = \partial_{s=0}\Gamma$, which is the variation field of $\Gamma$. Computing $J(1)$ we obtain

$$J(1) = \partial_{s=0}\Gamma(s,1) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(v+sw) = d_v(\exp_p)(w) = 0,$$

which shows that the Jacobi field $J$ vanishes at the endpoints.

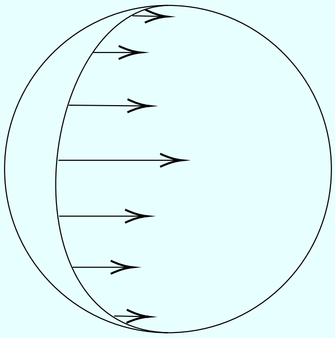
($\Leftarrow$) Conversely, if $p$ and $q$ are conjugate, there exists a Jacobi field $J \in \mathscr{J}(\gamma) \setminus \{0\}$ that vanishes at the endpoints, i.e. $J(0) = J(1) = 0$. By Lemma 2.1, $J$ is the variation field of

$$\Gamma(s,t) = \exp_p\big(t(v+sw)\big),$$

where $D_t J(0) = w$. But since $J(1) = d_v(\exp_p)(w)$ (as above), we obtain

$$d_v(\exp_p)(w) = 0.$$

□



# CHAPTER 4

# The Second Variation of Length

## 4.1 A Few Words on Variational Problems

Before referring to the second variation of the length functional, it is useful to mention the more general framework, namely the calculus of variations. Many mathematical problems, and especially those arising from physics or differential equations, are treated using techniques involving nonlinear functionals.[1]

**Example 4.1** We have already encountered the problem of minimizing length through the study of **geodesics**, which is in essence a variational problem. Given any suitable variation $\Gamma : J \times I \to M$, we know that on a sufficiently small neighborhood the length functional $L$ is minimized, for fixed endpoints. That is, the functional

$$L_g(\gamma) = \int_a^b |\dot{\gamma}(t)|_g \, dt$$

is minimized in the class

$$\mathcal{A} = \{\gamma : [a, b] \to M \mid \gamma \text{ a curve in } M \text{ with endpoints } \gamma(a) = p,\ \gamma(b) = q\}$$

provided the endpoints $p, q$ are sufficiently close. Other favorite problems among mathematicians working in the calculus of variations are the so-called **isoperimetric problems**. An example of an isoperimetric problem is the following: given a positive number $\mu$, can one find a simple closed curve of length $\mu$ that maximizes the area enclosed by it? That is, the question is whether the area functional

$$A(\gamma) = \int_{\text{int}(\gamma)} \sqrt{\det(g_{i,j})_{i,j}} \, dx \wedge dy$$

attains a maximum in the class

$$\mathcal{A} = \{\gamma \mid \gamma \text{ simple closed curve with } L_g(\gamma) = \mu\}.$$

Finally, there is also the (nonparametric) **Plateau problem** (and its more general version due to Douglas), in which one asks whether, given a map $\gamma : \partial\Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^m$, there exists an $n$-dimensional graph whose boundary contains $\gamma$ and which minimizes volume (or at least is a critical point). That is, we seek graphs $f : \Omega \to \mathbb{R}^n$ in $\mathbb{R}^{n+m}$ that minimize the Hausdorff measure

$$\mathscr{H}^n(\text{Gr}(f)) = \int_\Omega \sqrt{1 + \sum |\mathcal{M}\nabla f|^2} \, dx$$

[1] The term “nonlinear functional” coincides here with “not necessarily linear”.

within a suitable class $\mathcal{A}$. By $\mathcal{M}\nabla f$ we denote the $n \times n$ minors of $\nabla f$, and the sum is taken over all such minors. A problem related to minimal surfaces is the formation of alloys and the study of the Allen–Cahn equation $\varepsilon^2 \Delta u - \dot{W}(u) = 0$, whose solutions are the critical points of the energy functional

$$E(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\, dx.$$

This equation describes the formation of alloys and their interfaces. As $\varepsilon$ becomes small, solutions of the equation approach a minimal surface.
Perhaps the above examples give an idea of how fundamental the problems addressed by the calculus of variations are. What interests us as a first step is to describe some basic tools of the subject, which we will later compare with our results in differential geometry. We note for what follows that all our functions belong to a normed space.

**Definition 4.1 (Derivatives / First and second variations of functionals).** Let $J : \mathcal{A} \to \mathbb{R}$ be a functional. The **derivative** or **first variation** of $J$ at $f$, in the direction $h$, is defined by

$$\delta J(f, h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(f + \varepsilon h).$$

Correspondingly, the **second derivative** or **second variation** is defined by

$$\delta^2 J(f, h) = \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} J(f + \varepsilon h).$$

The directions $h$ are often also referred to as variations.

**Remark 4.1** Let $J : \mathcal{A} \to \mathbb{R}$ be a functional. If $J$ has a local extremum at $f_0$, then

$$\delta J(f_0, h) = 0, \quad \text{for every admissible variation } h,$$

that is, for every $h$ such that $f_0 + h \in \mathcal{A}$. Moreover, if there is a local minimum, then

$$\delta^2 J(f_0, h) \geqslant 0,$$

and if there is a local maximum, then

$$\delta^2 J(f_0, h) \leqslant 0.$$

*Proof.* Indeed, the function of $\varepsilon$, $J(f_0 + \Diamond h) : \mathbb{R} \to \mathbb{R}$, must have a local extremum at $\varepsilon = 0$, and therefore

$$\delta J(f_0, h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(f_0 + \varepsilon h) = 0.$$

The statements concerning the second variation follow again from the one-dimensional case. □

If one wishes to study problems on spaces that are Riemannian manifolds, there may not be such a direct description of a variation as in the case of functions $h$. In the example with curves, what would the expression $\gamma + \varepsilon h$ even mean? This is why we define variations carefully. We recall that an (admissible) variation of a curve $\gamma : [a, b] \to M$ is a continuous one-parameter family

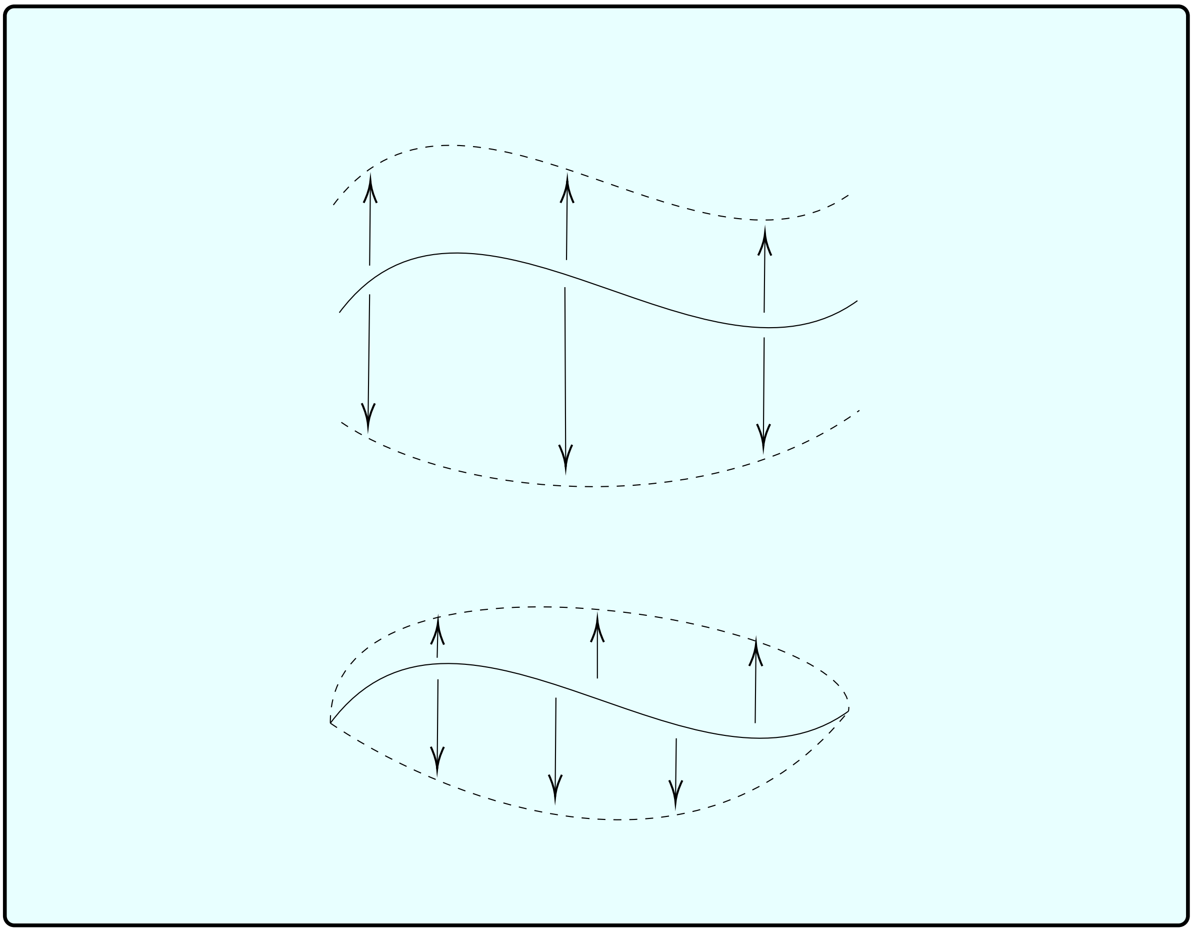
$$\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M, \qquad \Gamma(s, t) = \Gamma_s(t),$$

with the following properties:

- There exists a partition $\{a_0 < a_1 < \cdots < a_k\}$ of $[a, b]$ such that on each rectangle $(-\varepsilon, \varepsilon) \times [a_j, a_{j+1}]$ the map $\Gamma$ is smooth.
- Each $\Gamma_s : [a, b] \to M$ is piecewise regular and $\mathscr{C}^\infty$.
- We have $\Gamma_0(t) = \gamma(t)$ for all $t \in [a, b]$.

For variations we also define variation vector fields, i.e. vector fields $V : (-\varepsilon, \varepsilon) \times [a, b] \to TM$ with $V(s, t) \in T_{\Gamma(s,t)}M$. We are especially interested in the variation field along $\gamma$, namely

$$V(t) = \partial_{s=0}\Gamma(s, t).$$



*In the above figure: A variation and a normal variation are depicted.*

With the above definition in place, if we are interested in curves, it is natural to define the first and second variation of a functional as follows:

**Definition 4.2 (First and second variation — geometric version).** Let $(M, g)$ be a Riemannian manifold, let $\gamma : I \to M$ be a curve, and let $\Gamma : (-\varepsilon, \varepsilon) \times I \to M$ be a variation of $\gamma$. Let also $J : \mathcal{A} \to \mathbb{R}$ be a functional (on some class $\mathcal{A}$). We define the **first variation** (understood at $\gamma$) in the direction of $\Gamma$ by

$$\delta J(\gamma, \Gamma) = \delta J(\Gamma) = \frac{d}{ds}\Big|_{s=0} J(\Gamma_s),$$

and likewise the **second variation** by

$$\delta^2 J(\gamma, \Gamma) = \delta^2 J(\Gamma) = \frac{d^2}{ds^2}\Big|_{s=0} J(\Gamma_s).$$

**Remark 4.2** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a curve. Let also $J : \mathcal{A} \to \mathbb{R}$ be a functional (on some class $\mathcal{A}$). If $J$ has a local extremum at $\gamma$, then

$$\delta J(\Gamma) = 0$$

for every variation $\Gamma$ of $\gamma$. Moreover, if $\gamma$ is a minimizer, then

$$\delta^2 J(\Gamma) \geqslant 0,$$

whereas if it is a maximizer, then

$$\delta^2 J(\Gamma) \leqslant 0.$$

*Proof.* Indeed, the function of $s$, $J(\Gamma_s) : \mathbb{R} \to \mathbb{R}$, must have a local extremum at $s = 0$, and therefore

$$\delta J(\Gamma) = \frac{d}{ds}\Big|_{s=0} J(\Gamma_s) = 0.$$

The statements about the second variation again follow from the one-dimensional case. □

The above idea is simple, but very important and, if one thinks about it, quite deep. Instead of studying a problem in an awkward space (a space of functions), one studies many (in general infinitely many) easy problems on $\mathbb{R}$.

**Motivation 4.1** What we aim for, and what is suggested by Remarks 4.1 and 4.2, is the connection between first and second variations and minimization/maximization of functionals; in particular, we are interested in results related to minimizing length. It is already known, in the case where $\gamma$ is a geodesic, that locally we have $\delta L_g(\Gamma) = 0$ and conversely (so the condition $\delta L_g(\Gamma) = 0$ for every variation is necessary and sufficient for the existence of a local minimum). This result is described by saying that "geodesics locally minimize length". Later we will find a condition for a geodesic $\gamma$ which guarantees $\delta^2 L_g(\Gamma) < 0$ when the interval of computation becomes very large. Consequently, this shows that geodesics on large intervals do not necessarily minimize length.

Of course, variations have other applications beyond these basic results, such as the Morse, Bonnet–Myers, and Synge–Weinstein theorems.

## 4.2 The Second Variation Formula

**Theorem 4.1** (The second variation formula)**.** *Let $(M, g)$ be a Riemannian manifold. Let $\gamma : [a, b] \to M$ be a unit-speed geodesic, and let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a normal variation (i.e. $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$). Let $V$ be the corresponding variation field. Then*

$$\delta^2 L_g(\Gamma) = \frac{d^2}{ds^2}\Big|_{s=0} L_g(\Gamma_s) = \int_a^b |D_t V^\perp|^2 - \mathrm{Rm}(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\, dt,$$

*where $V^\perp$ is the normal component of $V$.*

*Proof.* In the proof we denote $T = \partial_t \Gamma$ and $S = \partial_s \Gamma$, and we also take a partition $\{a_0 < a_1 < \cdots < a_k\}$ as in the definition of variations. Differentiating once we have

$$\frac{d}{ds} L_g(\Gamma_s|_{[a_{j-1},a_j]}) = \frac{\partial}{\partial s} \int_{a_{j-1}}^{a_j} \langle T, T\rangle^{1/2}\, dt = \int_{a_{j-1}}^{a_j} \frac{\partial}{\partial s} \langle T, T\rangle^{1/2}\, dt,$$

where we interchange differentiation and integration since sufficient smoothness has been assumed. Differentiating the inner product gives

$$\int_{a_{j-1}}^{a_j} \frac{\partial}{\partial s} \langle T, T\rangle^{1/2}\, dt = \int_{a_{j-1}}^{a_j} \frac{1}{2} \langle T, T\rangle^{-1/2} 2\langle D_s T, T\rangle\, dt,$$

and by the Symmetry Lemma,

$$\int_{a_{j-1}}^{a_j} \frac{1}{2} \langle T, T\rangle^{-1/2} 2\langle D_s T, T\rangle\, dt = \int_{a_{j-1}}^{a_j} \frac{\langle D_t S, T\rangle}{\langle T, T\rangle^{1/2}}\, dt.$$

We can differentiate the last identity once more to obtain the second variation:

$$\begin{aligned} \frac{d^2}{ds^2} L_g(\Gamma_s|_{[a_{j-1},a_j]}) &= \int_{a_{j-1}}^{a_j} \frac{\partial}{\partial s} \frac{\langle D_t S, T\rangle}{\langle T, T\rangle^{1/2}}\, dt \\ &= \int_{a_{j-1}}^{a_j} \frac{\langle D_s D_t S, T\rangle}{\langle T, T\rangle^{1/2}} + \frac{\langle D_t S, D_s T\rangle}{\langle T, T\rangle^{1/2}} - \frac{\langle D_t S, T\rangle \langle D_s T, T\rangle}{\langle T, T\rangle^{3/2}}\, dt. \end{aligned}$$

(again interchanging differentiation and integration). In the first term we use the relation $D_s D_t S - D_t D_s S = \mathrm{R}(\partial_s \Gamma, \partial_t \Gamma) S$, and in the other two terms we use the Symmetry Lemma. Then

$$[...] = \int_{a_{j-1}}^{a_j} \frac{\langle D_t D_s S + \mathrm{R}(S, T)S, T\rangle}{\langle T, T\rangle^{1/2}} + \frac{\langle D_t S, D_t S\rangle}{\langle T, T\rangle^{1/2}} - \frac{\langle D_t S, T\rangle^2}{\langle T, T\rangle^{3/2}}\, dt.$$

Hence if $s = 0$, $\langle T, T\rangle = 1$,

$$\frac{d^2}{ds^2}\Big|_{s=0} L_g(\Gamma_s|_{[a_{j-1},a_j]}) = \left[ \int_{a_{j-1}}^{a_j} \langle D_t D_s S, T\rangle - \mathrm{Rm}(S, T, T, S) + |D_t S|^2 - \langle D_t S, T\rangle^2\, dt \right]_{s=0}.$$

But at $s = 0$ we have $D_t T = D_t \dot\gamma = 0$ since $\gamma$ is a geodesic, which allows us to write the sum of the first terms as follows:

$$\begin{aligned} \sum_{j=0}^{k} \left[ \int_{a_{j-1}}^{a_j} \langle D_t D_s S, T\rangle\, dt \right]_{s=0} &= \sum_{j=0}^{k} \left[ \int_{a_{j-1}}^{a_j} \frac{d}{dt} \langle D_s S, T\rangle\, dt \right]_{s=0} \\ &= \sum_{j=0}^{k} \left[ [\langle D_s S, T\rangle]_{a_{j-1}}^{a_j} \right]_{s=0} \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $D_{s=0} S = D_{s=0} \partial_s \Gamma = 0$ at $a_0 = a$ and $a_k = b$ (since there is no variation at the endpoints, by normality of $\Gamma$). Thus the first term vanishes and we obtain the simpler relation:

$$\begin{aligned} \frac{d^2}{ds^2}\Big|_{s=0} L_g(\Gamma_s) &= \sum_{j=0}^{k} \left[ \int_{a_{j-1}}^{a_j} -\mathrm{Rm}(S, T, T, S) + |D_t S|^2 - \langle D_t S, T\rangle^2\, dt \right]_{s=0} \\ &= \int_a^b -\mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) + |D_t V|^2 - \langle D_t V, \dot\gamma\rangle^2\, dt. \end{aligned}$$

These are essentially the main computations. It remains to write $V = V^\perp + V^\top$, where $V^\top = \langle V, \dot\gamma\rangle\dot\gamma$. Then

$$(D_tV)^\top = \langle D_tV, \dot\gamma\rangle\dot\gamma = D_t\langle V, \dot\gamma\rangle\,\dot\gamma = D_tV^\top, \text{ since } D_t\dot\gamma = 0$$

(and similarly $(D_tV)^\perp = D_tV^\perp$), hence

$$|D_tV|^2 = |(D_tV)^\top|^2 + |(D_tV)^\perp|^2 = \langle D_tV, \dot\gamma\rangle^2 + |D_tV^\perp|^2.$$

The first term cancels the corresponding term in the second variation formula we already found, and the second term is one of the desired ones. Finally, regarding the term $\mathrm{Rm}(V, \dot\gamma, \dot\gamma, V)$, by the symmetry

$$\mathrm{Rm}(\dot\gamma, \dot\gamma, \Diamond, \Diamond) = \mathrm{Rm}(\Diamond, \Diamond, \dot\gamma, \dot\gamma) = 0,$$

by the first Bianchi identity, and from the fact that $V^\top$ is parallel, it follows after some computations that

$$\mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) = \mathrm{Rm}(V^\perp, \dot\gamma, \dot\gamma, V^\perp)$$

(try to prove this). Combining everything above, we obtain the second variation formula $\delta^2 L_g(\Gamma)$:

$$\delta^2 L_g(\Gamma) = \frac{d^2}{ds^2}\Big|_{s=0} L_g(\Gamma_s) = \int_a^b |D_tV^\perp|^2 - \mathrm{Rm}(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\,dt.$$

□

It is clear from the second variation formula that what plays an essential role is the normal component of the variation field. This should be intuitively evident, since a tangential variation contributes only to reparametrizations of $\gamma$. From now on, there is no need to deal with general variation fields, but rather with normal variation fields. **We restrict our study to normal variation fields**, that is, to fields $V$ along $\gamma$ for which $V = V^\perp$.

Theorem 4.1 naturally leads to the definition of the index form of $\gamma$.

**Definition 4.3 (The index form).** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a unit-speed geodesic. For $V, W \in \mathscr{X}(\gamma)$ we define the **index form** of $\gamma$ by

$$I(V, W) = \int_a^b \langle D_tV, D_tW\rangle - \mathrm{Rm}(V, \dot\gamma, \dot\gamma, W)\,dt.$$

Theorem 4.1, together with the properties of the second variation in Remark 4.2, yields the following remark.

**Remark 4.3** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a unit-speed geodesic. If $\Gamma$ is a normal variation and $V$ is the corresponding variation field, then the minimizing property of $\gamma$ implies

$$I(V, V) \geqslant 0.$$

This is the analogue of the general result for functionals: if $J : \mathcal{A} \to \mathbb{R}$ is minimized, then $\delta^2 J(\Gamma) \geqslant 0$.

The formula for $I$ on Jacobi fields can take a specific simplified form, essentially given by the following proposition.

**Proposition 4.1.** *Let $(M,g)$ be a Riemannian manifold and let $\gamma : [a,b] \to M$ be a geodesic. For every piecewise smooth vector fields $V, W$ along $\gamma$,*

$$I(V,W) = -\int_a^b \langle D_t^2 V + \mathrm{R}(V,\dot\gamma)\dot\gamma, W\rangle\, dt + [\langle D_t V, W\rangle]_a^b - \sum_{j=1}^{k-1} \langle \Delta_j D_t V,\ W(a_j)\rangle.$$

*Here $\{a_0 < a_1 < \cdots < a_k\}$ denotes a partition such that $V, W$ are smooth on each subinterval, and $\Delta_j$ is the jump operator $D_{t=a_j^+}V - D_{t=a_j^-}V$.*

*Proof.* We work on the subintervals $[a_{j-1}, a_j]$. Differentiating $\langle D_t V, W\rangle$ gives

$$\frac{d}{dt}\langle D_t V, W\rangle = \langle D_t^2 V, W\rangle + \langle D_t V, D_t W\rangle,$$

hence integrating the last term yields

$$\int_{a_{j-1}}^{a_j} \langle D_t V, D_t W\rangle\, dt = -\int_{a_{j-1}}^{a_j} \langle D_t^2 V, W\rangle\, dt + [\langle D_t V, W\rangle]_{a_{j-1}}^{a_j}.$$

Summing these identities and subtracting the term $\int_a^b \mathrm{Rm}(V,\dot\gamma,\dot\gamma,V)\, dt$ gives the desired formula. □

A consequence of the previous proposition is the following remark.

**Remark 4.4** Let $(M,g)$ be a Riemannian manifold. If $\gamma$ is a geodesic and $V$ is a normal, piecewise smooth vector field along $\gamma$, then $I(V,W) = 0$ for all normal, piecewise smooth vector fields $W$ along $\gamma$ if and only if $V$ is a Jacobi field.

*Proof.* One direction is immediate, namely under the assumption that $V$ is a Jacobi field. Jacobi fields are solutions of a linear differential equation, so the existence theorem for solutions implies (among other things) smoothness. For the other direction, the proof has significant similarities with the argument showing that minimal surfaces satisfy a geodesic-type equation, so we will be brief.

On any interval of the form $[a_{j-1}, a_j]$, choose a bump function $\varphi(t)$, and also consider the normal field

$$W = \varphi(t)\left(D_t^2 V + \mathrm{R}(V,\dot\gamma)\dot\gamma\right).$$

Since $\varphi$ is a bump function, Proposition 4.1 implies

$$0 = I(V,W) = -\int_{a_{j-1}}^{a_j} \varphi(t)\left|D_t^2 V + \mathrm{R}(V,\dot\gamma)\dot\gamma\right|^2 dt.$$

Moreover, this holds for every $\varphi \in \mathscr{C}_c^\infty([a_{j-1}, a_j])$, which shows that

$$D_t^2 V + \mathrm{R}(V,\dot\gamma)\dot\gamma = 0 \quad \text{on } [a_{j-1}, a_j],$$

i.e. $V$ is Jacobi on each subinterval. It remains to show that there are no "corners" at the junction points $a_j$. Choose a field $W$ such that

$$W(a_j) = D_{t=a_j^+}V - D_{t=a_j^-}V, \quad j \in \{1,\ldots,k-1\}, \qquad W(a) = W(b) = 0.$$

Then again by Proposition 4.1, together with the fact that $V$ is Jacobi on each piece,

$$0 = I(V, W) = -\sum_{j=1}^{k-1} |\Delta_j D_t V|^2,$$

that is, $\Delta_j D_t V = 0$ for every $j$. This ensures there are no corners, and hence the desired conclusion (recall also the uniqueness of Jacobi fields). □

## 4.3 The second variation of energy

Another formulation of the previous results, which is more convenient in certain places, is the one involving energy. In [1], for example, the variation results are presented entirely in terms of energy.

**Definition 4.4 (Kinetic energy).** Let $(M, g)$ be a Riemannian manifold, $\gamma : [a, b] \to M$ a curve, and $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ a variation of $\gamma$. We define the **kinetic energy** of the $s$–variation of $\gamma$ by

$$E(s) = \int_a^b |\partial_t \Gamma(s, t)|^2 \, dt.$$

It is not difficult to show that, if $\gamma$ is a geodesic, then for every normal variation $\Gamma$ we have $E(\gamma) \leqslant E(s)$. Following also the proof of the first variation of length, one can show that there is an analogous formula for the first variation of the energy. However, here we will be concerned only with the second variation of energy.

Combining the proofs of Theorem 4.1 and Proposition 4.1, we obtain the following theorem:

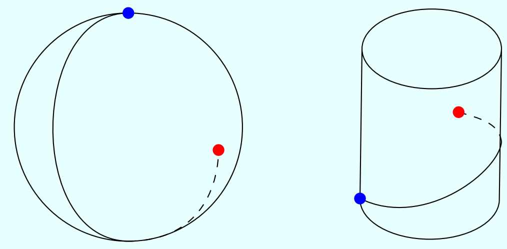
**Theorem 4.2** (The second variation of energy)**.** *Let $(M, g)$ be a Riemannian manifold, $\gamma : [a, b] \to M$ a geodesic, and $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ a normal variation. Then*

$$\frac{1}{2}\ddot{E}(0) = -\int_a^b \langle D_t^2 V + \mathrm{R}(V, \dot{\gamma})\dot{\gamma}, V \rangle \, dt - \sum_{j=1}^{k-1} \langle \Delta_j D_t V, V(a_j) \rangle.$$

*Here $V$ denotes the variation field of $\Gamma$, and $\{a_0 < a_1 < \cdots < a_k\}$ is a partition such that $V$ is smooth on each subinterval.*

## 4.4 Failure of minimizing beyond conjugate points

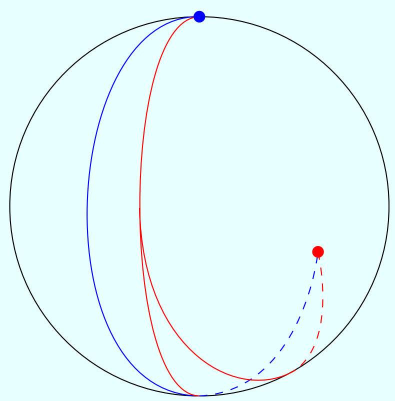
We know that, locally, geodesics are length-minimizing curves. Globally, however, as is seen from the example of the sphere or the cylinder, geodesics do not minimize length on arbitrarily large domains.



**Motivation 4.2** We are interested in investigating how large this domain must be for the geodesic to cease being minimizing. Taking into account that $\delta^2 L_g(\Gamma) \geqslant 0$ when $L_g$ is minimized, we might want to find conditions under which $\delta^2 L_g(\Gamma) < 0$. Also, since the critical points of $d_v(\exp_p)$ are related to conjugate points of $p$, we expect conjugacy to play a role as well.

**Definition 4.5 (Conjugate points along curves).** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, c] \to M$ be a geodesic. We say that $\gamma$ has a **conjugate** point if there exists $b \in (a, c]$ such that $\gamma(a)$ and $\gamma(b)$ are conjugate. The conjugate point is called **interior** if $b \in (a, c)$.

The geometric picture one has is the following: suppose that on the sphere $\mathbb{S}^2$ we take geodesics that pass beyond antipodal points (say $p = \gamma(a)$ and $q$ is the antipodal point). Then between $p$ and $q$ there is (for instance by Proposition 1.1) an entire variation through geodesics of $\gamma$ (here, the semicircles between the antipodal points). Choose one semicircle $C$ (different from the segment of $\gamma$), and observe that the curve $C$ together with the final segment of $\gamma$ is a piecewise geodesic curve of the same length as $\gamma$. At the point $q$ there is a corner, so by the usual smoothing arguments for corners of geodesics we can produce a new curve of shorter length connecting the endpoints of $\gamma$.



The following theorem essentially provides a condition under which geodesics do not minimize length.

**Theorem 4.3.** *Let $(M, g)$ be a Riemannian manifold and let $p, q \in M$. If $\gamma$ is a geodesic between $p$ and $q$ with an interior conjugate point, then there exists a normal field $V \in \mathscr{X}(\gamma)$ with $I(V, V) < 0$. In other words, $\delta^2 L_g < 0$ (for a suitable variation).*

*Proof.* Consider a geodesic $\gamma : [a, c] \to M$ with $\gamma(a) = p$, $\gamma(c) = q$, and an interior conjugate point $\gamma(b)$ with $b \in (a, c)$. Since $\gamma(a)$ and $\gamma(b)$ are conjugate, there exists a Jacobi field vanishing at $a$ and $b$, say $J$. Define

$$Y(t) = \begin{cases} J(t), & t \in [a, b], \\ 0, & t \in [b, c]. \end{cases}$$

and note that $Y$ is a normal and piecewise smooth vector field along $\gamma$. At $t = b$ there may be a "corner", and for this reason we also define a smooth vector field $W$ along $\gamma$ such that

$$W(b) = \Delta_{t=b} D_t Y \quad \text{and } W \text{ has compact support.}$$

In fact there is always a jump in the derivative, because if $0 = \Delta_{t=b} D_t Y = -D_{t=b} J$, then by uniqueness $J$ would be identically zero. Also, for small $\varepsilon > 0$ define $V_\varepsilon = Y + \varepsilon W$, and we have

$$I(V_\varepsilon, V_\varepsilon) = I(Y + \varepsilon W, Y + \varepsilon W) = I(Y, Y) + 2\varepsilon I(Y, W) + \varepsilon^2 I(W, W).$$

The field $Y$ satisfies the Jacobi equation on $[a, b]$ and on $[b, c]$, with $Y(b) = 0$, and hence by Proposition 4.1,

$$I(Y, Y) = -\langle \Delta_{t=b} D_t Y, Y(b) \rangle = 0.$$

Similarly,

$$I(Y, W) = -\langle \Delta_{t=b} D_t Y, W(b) \rangle = -|W(b)|^2.$$

Therefore,

$$I(V_\varepsilon, V_\varepsilon) = -2\varepsilon |W(b)|^2 + \varepsilon^2 I(W, W) = O(-\varepsilon),$$

so for $\varepsilon = \varepsilon_0$ sufficiently small we get $I(V_{\varepsilon_0}, V_{\varepsilon_0}) < 0$. Choose $V = V_{\varepsilon_0}$. □

We can also prove a result that looks like a kind of converse to the previous one. Namely, we can show that if there are no conjugate points, then the curve $\gamma$ is minimizing among nearby normal variations.

**Lemma 4.1** Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a geodesic. If $J_1, J_2 \in \mathscr{J}(\gamma)$, then the quantity

$$f(t) = \langle D_t J_1, J_2 \rangle(t) - \langle J_1, D_t J_2 \rangle(t)$$

is constant.

*Proof.* It is a matter of computations, differentiating $f$. □

**Theorem 4.4.** *Let $(M, g)$ be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a geodesic with no interior conjugate points. If $V$ is any normal, piecewise smooth vector field along $\gamma$, then $I(V, V) \geqslant 0$. Equality holds if and only if $V$ is a Jacobi field. In particular, if $\gamma(b)$ is not conjugate to $\gamma(a)$, then $I(V, V) > 0$.*

*Proof.* The proof is lengthy, so we present it briefly in steps.

- We may assume without loss of generality that $a = 0$. Set $p = \gamma(0)$ and let $\{w_1, \ldots, w_n\}$ be an orthonormal basis of $T_p M$ with $w_1 = \dot\gamma(0)$. For each $i \geqslant 2$, let $J_i$ denote the Jacobi field with $J_i(0) = 0$ and $D_{t=0} J_i = w_i$.

- Since $\gamma$ has no conjugate points, no nontrivial linear combination of the $J_i$ vanishes. Hence (for instance by a dimension argument) the fields $J_i$ span, for each $t$, the space $T^{\perp}_{\dot\gamma(t)} T_{\gamma(t)} M$. Therefore, for $V$ as in the statement we can write

$$V(t) = u^i(t) J_i(t)$$

for smooth functions $u^i : (0, b) \to \mathbb{R}$.

- In normal coordinates, from the form of Jacobi fields vanishing at a point, we can write
$$J_i(t) = t\frac{\partial}{\partial x_i}\Big|_{\gamma(t)} \quad \text{and} \quad V(t) = tu^i(t)\frac{\partial}{\partial x_i}\Big|_{\gamma(t)}.$$
In this form one can show that the $u^i$ extend smoothly to $[0, b]$ (we omit this).

- Define
$$X = u^i D_t J_i \quad \text{and} \quad Y = \dot{u}^i J_i,$$
and observe that $D_t V = X + Y$ on the intervals where $V$ is smooth. Let $\{a_0 < a_1 < \cdots < a_k\}$ be the corresponding partition.

- We aim to show that
$$|D_t V|^2 - \mathrm{Rm}(V, \dot\gamma, \dot\gamma, V) = \frac{d}{dt}\langle V, X\rangle + |Y|^2.$$
Start by differentiating $\langle V, X\rangle$:
$$\frac{d}{dt}\langle V, X\rangle = \langle D_t V, X\rangle + \langle V, D_t X\rangle = \langle X + Y, X\rangle + \langle V, D_t X\rangle.$$
From the Jacobi equation one can show that
$$D_t X = \dot{u}^i D_t J_i - \mathrm{R}(V, \dot\gamma)\dot\gamma,$$
hence
$$\langle D_t X, V\rangle = \langle \dot{u}^i D_t J_i, u^\lambda J_\lambda\rangle - \mathrm{Rm}(V, \dot\gamma)\dot\gamma. \tag{4.1}$$
But at $t = 0$ we have $\langle D_t J_i, J_\lambda\rangle - \langle J_i, D_t J_\lambda\rangle = 0$, and therefore by Lemma 4.1, $\langle D_t J_i, J_\lambda\rangle = \langle J_i, D_t J_\lambda\rangle$. Thus the first term in (4.1) becomes
$$\langle \dot{u}^i D_t J_i, u^\lambda J_\lambda\rangle = \dot{u}^i u^\lambda \langle D_t J_i, J_\lambda\rangle = \langle \dot{u}^i J_i, u^\lambda D_t J_\lambda\rangle = \langle Y, X\rangle.$$
Substituting this into (4.1) and then into the derivative formula above gives the desired identity.

- Now compute:
$$\begin{aligned} I(V, V) &= \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} \big(|D_t V|^2 - \mathrm{Rm}(V, \dot\gamma, \dot\gamma, V)\big)\, dt \\ &= \sum_{j=1}^{k} [\langle V, X\rangle]_{a_{j-1}}^{a_j} + \int_0^b |Y|^2\, dt \\ &= \int_0^b |Y|^2\, dt, \end{aligned}$$
since $V$ and $X$ are continuous. Hence $I(V, V) \geqslant 0$.

- If $I(V, V) = 0$, then necessarily $\dot{u}^i \equiv 0$, i.e. the $u^i$ are constant. Therefore $V$ is an $\mathbb{R}$–linear combination of Jacobi fields, hence itself a Jacobi field.

□

A much more general theorem is the Morse index theorem, which can be found in [1]. The Morse index theorem justifies the terminology "index" in the definition of $I$, and it is important at least to mention it. The statement of the theorem is as follows:

**Theorem 4.5** (The Morse index theorem)**.** *Let* $(M, g)$ *be a Riemannian manifold and let* $\gamma : [a, b] \to M$ *be a geodesic. Denote by* $\mathcal{I}$ *the maximal dimension of subspaces of normal vector fields along* $\gamma$ *on which the index form* $I$ *is negative definite. Then* $\mathcal{I}$ *equals the number of conjugate points of* $\gamma$ *in* $(a, b)$*, counted with multiplicity (i.e. taking the order into account).*

CHAPTER 5

# Cut points

## 5.1 Cut times and cut loci

In this chapter we will study conjugate points further, and the diffeomorphism property of $\exp_p$.

**Motivation 5.1** Suppose we have a curve $\gamma : [a, b] \to M$ which is a unit-speed geodesic. There are two possibilities: either the geodesic is minimizing everywhere, or (the more common case) beyond some point it ceases to be minimizing (for example, this would happen if $\gamma$ passed through a conjugate point). By continuity, the restriction of $\gamma$ that is length-minimizing has domain either $[a, t_0)$ or $[a, \infty)$. In fact, because of unit speed, the number $t \in \{t_0 - a, \infty\}$ is the time until a moving point along $\gamma$ stops moving along a length-minimizing curve.

We recall that we use the notation $\gamma_v$ for the maximal geodesic curve starting at a point $p$ with initial velocity $v$. For convenience, we will also take the domain to be $[0, b]$.

**Definition 5.1 (Cut time and cut locus).** Let $(M, g)$ be a complete and connected Riemannian manifold, and let $p \in M$, $v \in T_pM$. We define the **cut time**:

$$t_{\text{cut}}(p, v) = \sup\{b > 0 \mid \text{the geodesic } \gamma_v|_{[0,b]} \text{ is minimizing}\}.$$

We also define the **cut locus**:

$$\text{Cut}(p) = \{q \in M \mid \exists \gamma_v \text{ with } \gamma_v(t_{\text{cut}}(p, v)) = q\},$$

whose elements are called **cut points**.

**Proposition 5.1.** *Let $(M, g)$ be a complete and connected Riemannian manifold, $p \in M$, and let $v \in T_pM$ be unit. Write $t_{\text{cut}} = t_{\text{cut}}(p, v)$. Then:*

(a) *If $0 < b < t_{\text{cut}}$, then $\gamma_v|_{[0,b]}$ has no conjugate points and is the unique unit-speed minimizing curve.*

(b) *If $t_{\text{cut}} < \infty$, then $\gamma_v|_{[0,t_{\text{cut}}]}$ is minimizing and: i) $\gamma_v(t_{\text{cut}})$ is conjugate to $p$ and/or ii) there exist at least two unit-speed minimizing geodesics from $p$ to $\gamma_v(t_{\text{cut}})$.*

*Proof.* For (a): Let $0 < b < t_{\text{cut}}$. By definition of $t_{\text{cut}}$, we can find $0 < b < c < t_{\text{cut}}$ such that $\gamma_v|_{[0,c]}$ is minimizing, and we observe that by Theorem 4.3 the point $\gamma_v(t)$ is not conjugate for any $0 < t \leqslant b$. Note that $\gamma_v|_{[0,b]}$ is minimizing since $\gamma_v|_{[0,c]}$ is (why?). The geodesic $\gamma_v|_{[0,b]}$ is also unique. If it were not (and there were another one, say $\gamma$), then by uniqueness of geodesics we would have $\dot\gamma(b) \neq \dot\gamma_v(b)$. Thus the curve $\zeta$ which equals $\gamma$ on $[0,b]$ and equals $\gamma_v$ on $[b,c]$ is a minimizing, piecewise geodesic curve with a corner. This means—as is well known—that it can be smoothed at the corner to produce a shorter curve, a contradiction.

For (b): Assume $t_{\text{cut}} < \infty$, and choose a sequence $\{b_j\}_{j=1}^\infty$ with $b_j \nearrow t_{\text{cut}}$, such that $\gamma_v|_{[0,b_j]}$ is minimizing. By continuity we have:

$$d_g(p, \gamma_v(t_{\text{cut}})) = \lim_{j\to\infty} d_g(p, \gamma_v(b_j)) = \lim_{j\to\infty} b_j = t_{\text{cut}},$$

which indicates that $\gamma_v$ is minimizing on $[0, t_{\text{cut}}]$. Now suppose that $\gamma_v(t_{\text{cut}})$ is not conjugate to $p$. We will show that necessarily there are two unit-speed minimizing geodesics from $p$ to $\gamma_v(t_{\text{cut}})$, and thus the proposition follows. Choose a sequence $\{b_j\}_{j=1}^\infty$ with $b_j \nearrow t_{\text{cut}}$, and by the definition of $t_{\text{cut}}$, none of the $\gamma_v|_{[0,b_j]}$ is minimizing. Hence, by Hopf–Rinow, we can find a unit-speed minimizing geodesic $\gamma_j : [0, a_j] \to M$ from $p$ to $\gamma_v(b_j)$, with $a_j < b_j$. Define the unit vectors $w_j = \dot\gamma_j(0)$. By compactness of the unit sphere, there is a convergent subsequence; without loss of generality assume $w_j \to w$. Similarly, since $\{a_j\}_{j=1}^\infty$ is bounded, we may assume (again WLOG) that $a_j \to a$. Then:

$$\gamma_v(b_j) = \gamma_j(a_j) = \exp_p(a_j w_j) \to \exp_p(aw),$$

and:

$$t_{\text{cut}} = d_g(p, \gamma_v(t_{\text{cut}})) = \lim_{j\to\infty} d_g(p, \gamma_j(a_j)) = \lim_{j\to\infty} a_j = a,$$

so $\gamma(t) = \exp_p(tw)$ on $[0, t_{\text{cut}}]$ is also a unit-speed minimizing geodesic. We must show that $\gamma_v$ and $\gamma$ are not the same. Since $\gamma_v(t_{\text{cut}})$ is not conjugate, Proposition 3.1 implies that $t_{\text{cut}}v$ is a regular point of $\exp_p$, hence $\exp_p$ is locally invertible there. Because $\exp_p(a_j w_j) = \exp_p(b_j v)$, and since $b_j v \to t_{\text{cut}} v$, by injectivity the vectors $a_j w_j$ must stay at some distance away from the neighborhood where $\exp_p$ is inverted. Passing to the limit gives $t_{\text{cut}} v \neq aw$, hence $\gamma_v$ and $\gamma$ do not coincide. □

We also state the following theorem, which we will not prove. We refer the reader to [4].

**Theorem 5.1** (Continuity of cut times)**.** *Let $(M,g)$ be a complete and connected Riemannian manifold. The function $t_{\text{cut}} : UTM \to (0,\infty]$ on the unit tangent bundle*

$$UTM = \{(p,v) \in TM \mid |v|_g = 1\}$$

*is continuous.*

## 5.2 The domain of injectivity

**Motivation 5.2** Our intuition, together with Proposition 5.1, indicates that geodesics behave poorly near cut times. Thus, in studying the exponential map it may be useful to study geodesics away from these times. This leads to the definition of the domain of injectivity.

**Definition 5.2 (Domain of injectivity and tangent cut locus).** Let $(M,g)$ be a complete and connected Riemannian manifold. We define the **domain of injectivity** of $p$ by

$$\text{ID}(p) = \{v \in T_pM \mid |v| < t_{\text{cut}}(p, v/|v|)\}.$$

We also define the boundary of this set, called the **tangent cut locus**:

$$\text{TCL}(p) = \partial\text{ID}(p) = \{v \in T_pM \mid |v| = t_{\text{cut}}(p, v/|v|)\}.$$

**Theorem 5.2.** *Let $(M, g)$ be a complete and connected Riemannian manifold, and let $p \in M$. Then:*

*(a)* $\mathrm{Cut}(p) \subseteq M$ *is a closed set of measure zero.*

*(b) The restriction of the exponential map to* $\overline{\mathrm{ID}(p)}$ *is surjective.*

*(c) The restriction of the exponential map to* $\mathrm{ID}(p)$ *is a diffeomorphism onto* $M \setminus \mathrm{Cut}(p)$.

*Proof.* For (a): Let $\{q_j\}_{j=1}^{\infty} \subseteq \mathrm{Cut}(p)$ be a sequence converging to $q$, and we will show that $q \in \mathrm{Cut}(p)$. Write $q_j = \exp_p(t_{\mathrm{cut}}(p, v_j)v_j)$ for unit vectors $v_j$. By compactness of the unit sphere, there is a convergent subsequence of $\{v_j\}$; without loss of generality assume $v_j \to v$. Then

$$t_{\mathrm{cut}}(p, v_j) \to t_{\mathrm{cut}}(p, v)$$

by continuity of $t_{\mathrm{cut}}$, and moreover $t_{\mathrm{cut}}(p, v) < \infty$. By continuity of $\exp_p$,

$$q_j = \exp_p(t_{\mathrm{cut}}(p, v_j)v_j) \to \exp_p(t_{\mathrm{cut}}(p, v)v) \in \mathrm{Cut}(p),$$

hence $q \in \mathrm{Cut}(p)$. As for the measure statement, the idea is to show that $\mathrm{TCL}(p)$ has measure zero. Then, since $\exp_p$ is smooth,

$$\mathrm{Cut}(p) = \exp_p(\mathrm{TCL}(p))$$

also has measure zero. The set $\mathrm{TCL}(p)$ is easily seen to have measure zero: represent $\mathrm{TCL}(p)$ locally as a graph coming from the implicit relation $|v| = t_{\mathrm{cut}}(p, v/|v|)$ (here one introduces coordinates, for example polar coordinates $r = t_{\mathrm{cut}}(p, \theta^1, \ldots, \theta^n)$). At most a countable collection of local graphs covers $\mathrm{TCL}(p)$, and each of these has measure $0$ (being the graph of a continuous function). Hence $\mathrm{TCL}(p)$ has measure $0$.

For (b): Use the Hopf–Rinow theorem.

For (c): From the definitions we have $\exp_p(\mathrm{ID}(p)) = M \setminus \mathrm{Cut}(p)$. Since $\exp_p(\mathrm{ID}(p))$ contains no cut points, it should contain no conjugate points of $p$ either (see Theorem 4.3). Since there are no cut points, $\exp_p$ on $\mathrm{ID}(p)$ is 1–1. Since there are no conjugate points, $\exp_p$ on $\mathrm{ID}(p)$ is a local diffeomorphism (by Proposition 3.1), and thus, combined with injectivity, a diffeomorphism. □

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